HHT: the theory, implementation and application

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What is frequency?

- Frequency definition
- Fourier glass
- Instantaneous frequency
- Signal composition: trend, periodical, stochastic, and discontinuity

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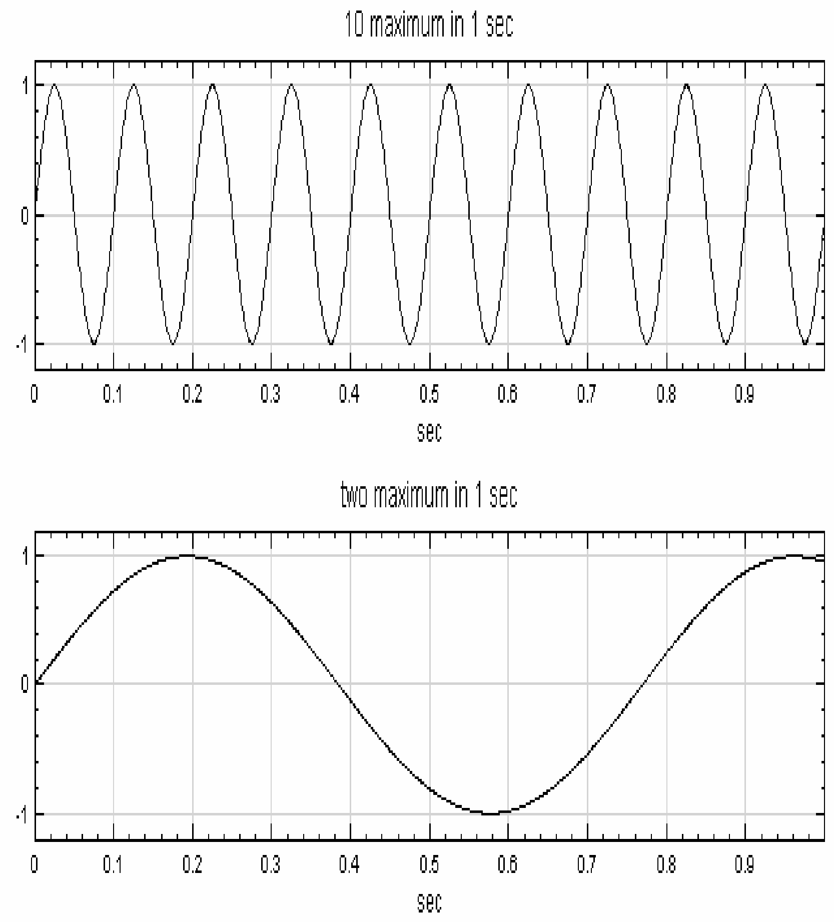
What is frequency?

- The inverse of period. But then what is period?
- Number of times of “happening” in a period. Then what is “happening”? Zero crossing, extrema, or others?
- Fourier transform.

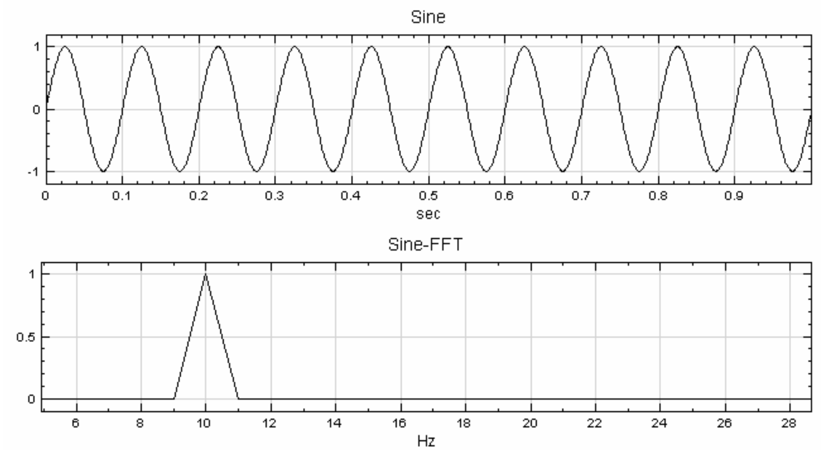
Frequency definition (1)

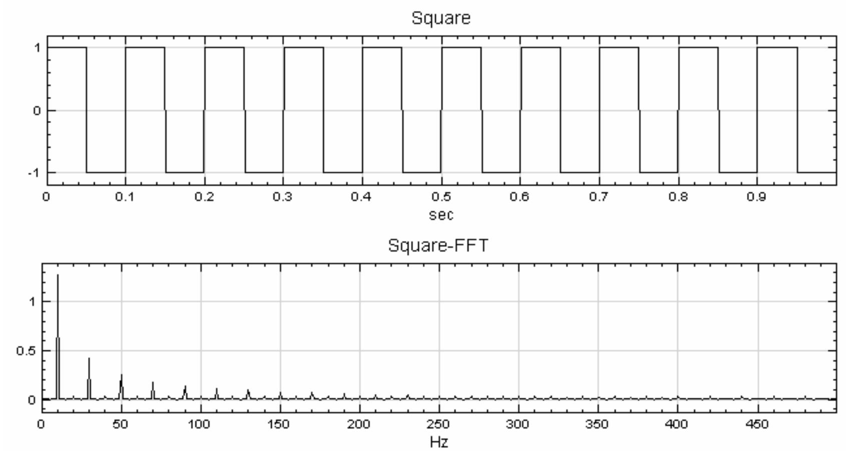
- Period is defined as number of events in duration of time. And frequency is the inverse of period.
- The concept of period is an average. So do the frequency.
- Since there is no instantaneous period, there is no instantaneous frequency.
- Putting on Fourier glasses, uncertainty principle comes into play for sinusoidal signal. That is, the lower the frequency to be identified, the longer the duration of signal.



Single frequency?

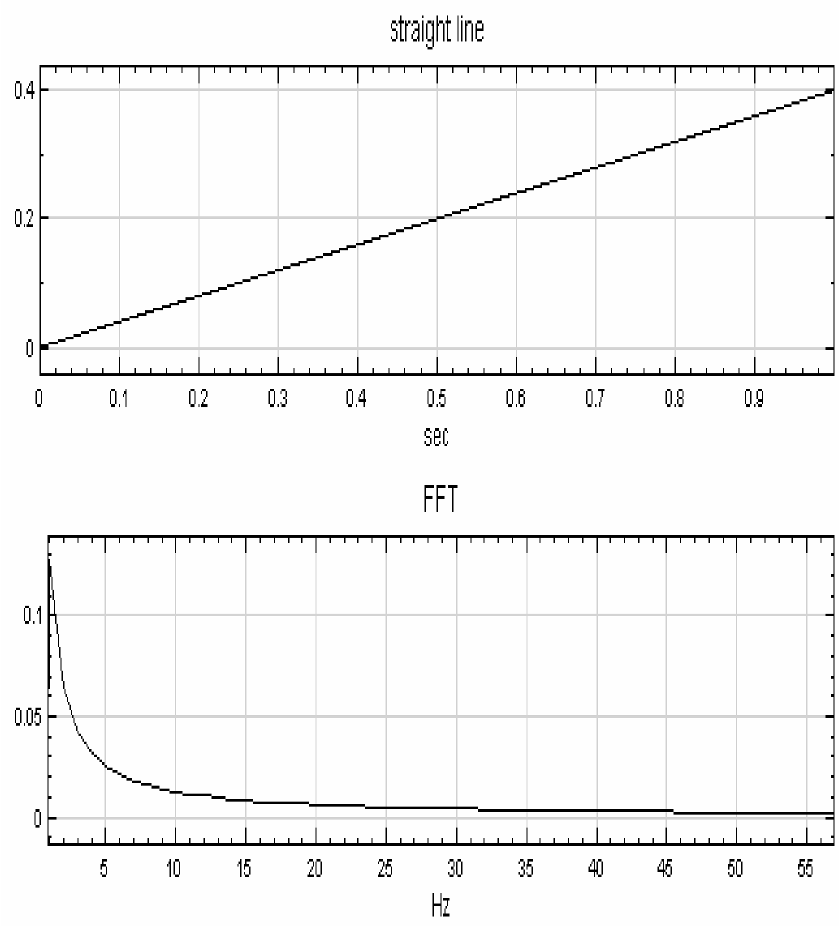
- Is the squared periodical wave of single frequency?
- With Fourier glasses, sine wave is of single frequency while squared wave is of multiple frequencies.
- What is the definition of frequency?
- What do we want to see?





Non-Periodical Signal of many frequencies?

- Does a straight has a frequency?
- Putting on Fourier glasses, we see so many frequencies from a straight line.
- Again what do we want to see?



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What do we want from frequency? Data perception

- Signal is normally composed of four parts: trend, periodical, discontinuity(jump, end effect), and stochastic.
- Trend is best perceived in time domain.
- Discontinuity which might result in infinitely many components in Fourier analysis is better eliminated before processing.
- Stochastic part currently is not involved in this discussion.
- Periodical signal can be well perceived in time-frequency-energy representation.

Why Fourier?

1. LTI system
2. Sine/Cosine: Eigen-function of LTI system
3. Frequency: Eigen-value of LTI system

Linear Time Invariant System

- Linear System :

$$\hat{L}(af_1(t)+bf_2(t)) = a\hat{L}f_1(t)+b\hat{L}f_2(t)$$

- Time Invariant:

$$\hat{L}f(t) = h(t)$$

$$\hat{L}f(t-u) = h(t-u)$$

LTI vs. Convolution

$$\hat{L}\delta(t) = G(t)$$

$$\hat{L}f(t) = \hat{L}\int f(u)\delta(u-t)du$$

$$= \int f(u)\hat{L}\delta(u-t)du$$

$$= \int f(u)G(u-t)du$$

$$= f * G$$

Exponential : eigen-function of LTI system

$$\hat{L}e^{jwt} = L(w)e^{jwt}$$

Sin and Cos functions are eigenfunction of LTI system.
Eigenvalue of a LTI system is frequency.

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Removal of Non-Periodical Signal

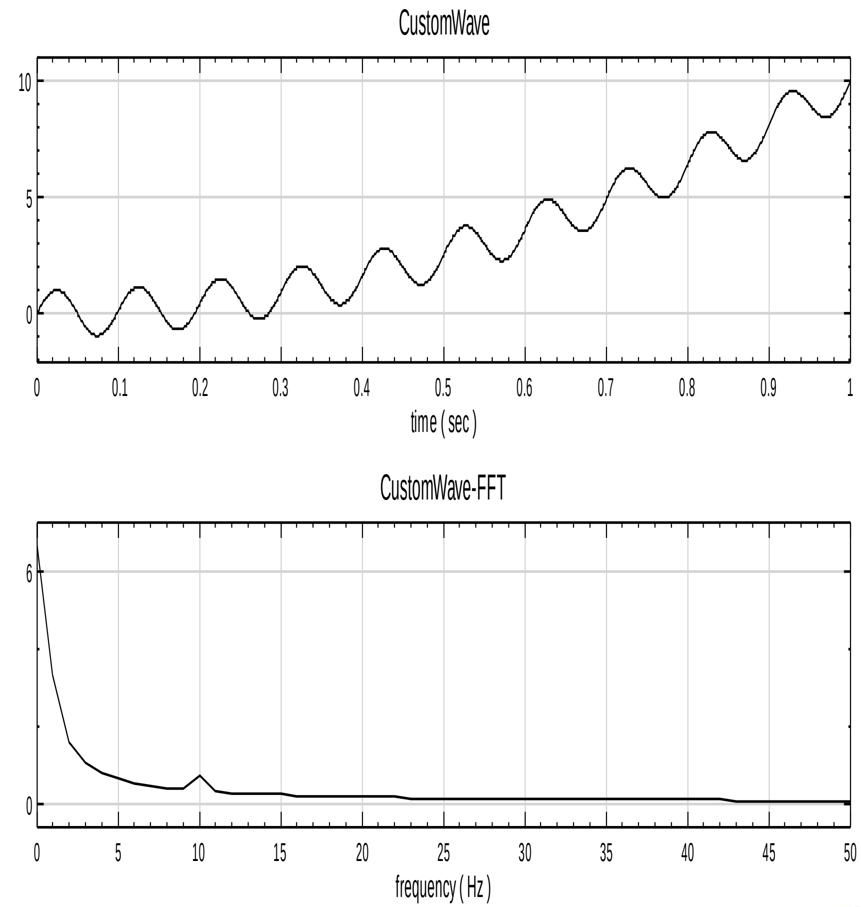
Frequency based filter

Iterative Gaussian Filter

EMD as filter

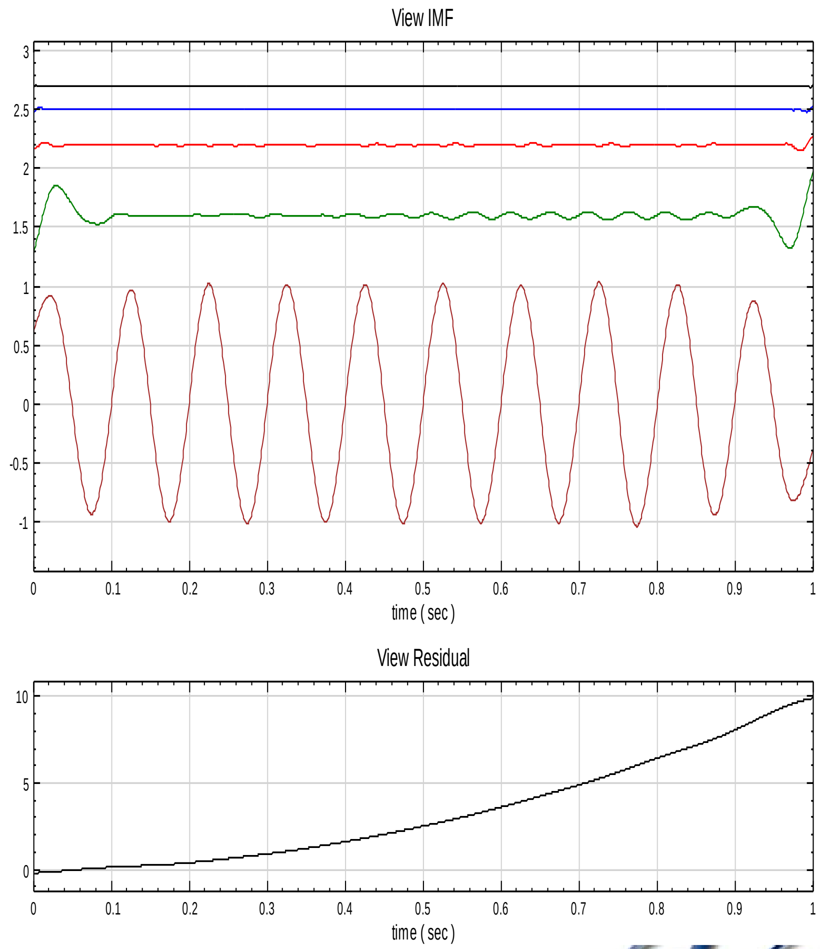
Trend embedded in signal of interest

- Contamination of non-periodical signal.



EMD for trend removal

- EMD is also a good way to separate trend signal.



Gaussian Filter

- Iterative moving least square method
- Zero phase error
- Fast algorithm developed by Prof. Cheng of NCKU.
- Continuous signal with trend can be written in the form:

$$x(t) = \sum_{k=1}^{n} a_k \cos \omega_k t + b_k \sin \omega_k t + \sum_{i=0}^{m} \alpha_i t^i$$

For each iteration of Gaussian filter, polynomial order m is reduced by 2.

- Gaussian function is the fundamental solution to heat equation. The idea of smoothing is similar to the mechanism of heat spreading, regarding signal as unbalanced temperature distribution to begin with.

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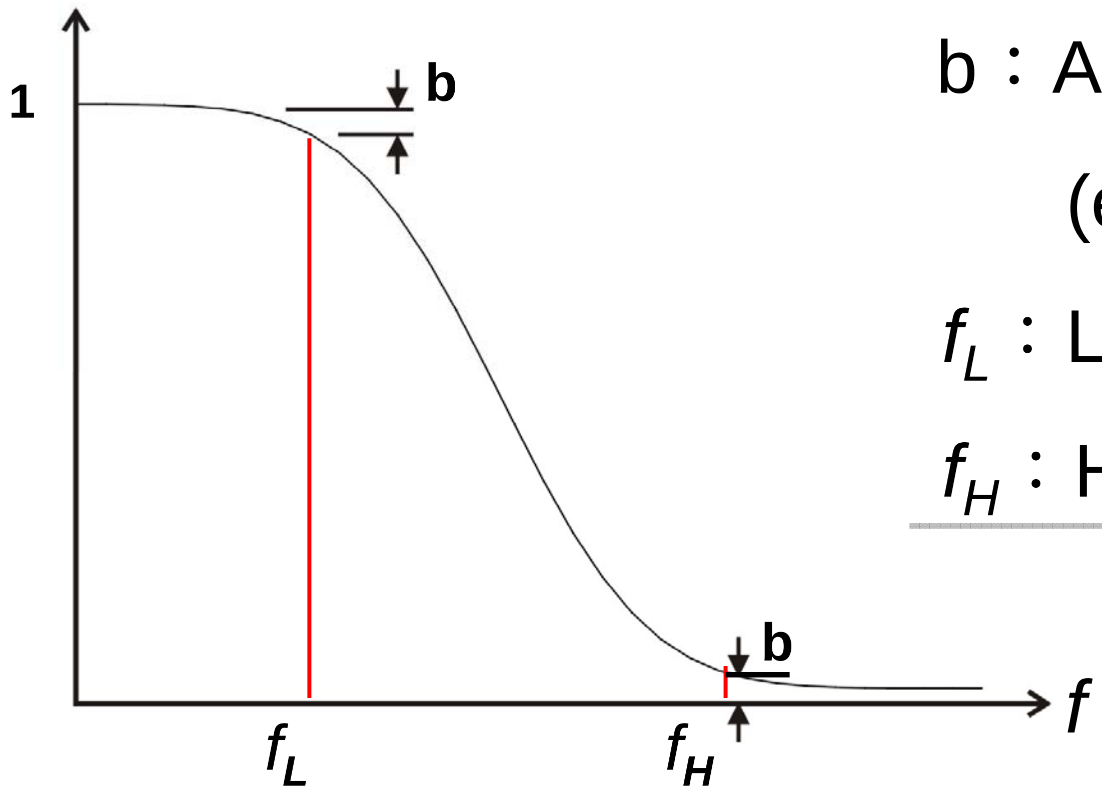
Iterative Gaussian Filter



b：Attenuation factor

(e.g. b=0.01)

f_L：Low frequency

f_H：High frequency

Iterative Gaussian Filter

- The Gaussian factor , σ , and number of iteration, *m*, are determined via solving

$$1-\left[1-e^{-2\pi^2(\sigma f_L)^2}\right]^m = b$$

$$1-\left[1-e^{-2\pi^2(\sigma f_H)^2}\right]^m = 1-b$$

Trend Estimator

T_C (Trend period) ：For low pass filter, period lower than the value will be annihilated. In practice Tc is approximately the length of “bump” (half wavelength) under which will be eliminated.

Cutoff frequency

$$f_C = \frac{2}{T_C}$$

$$f_L = \frac{2}{T_C}$$

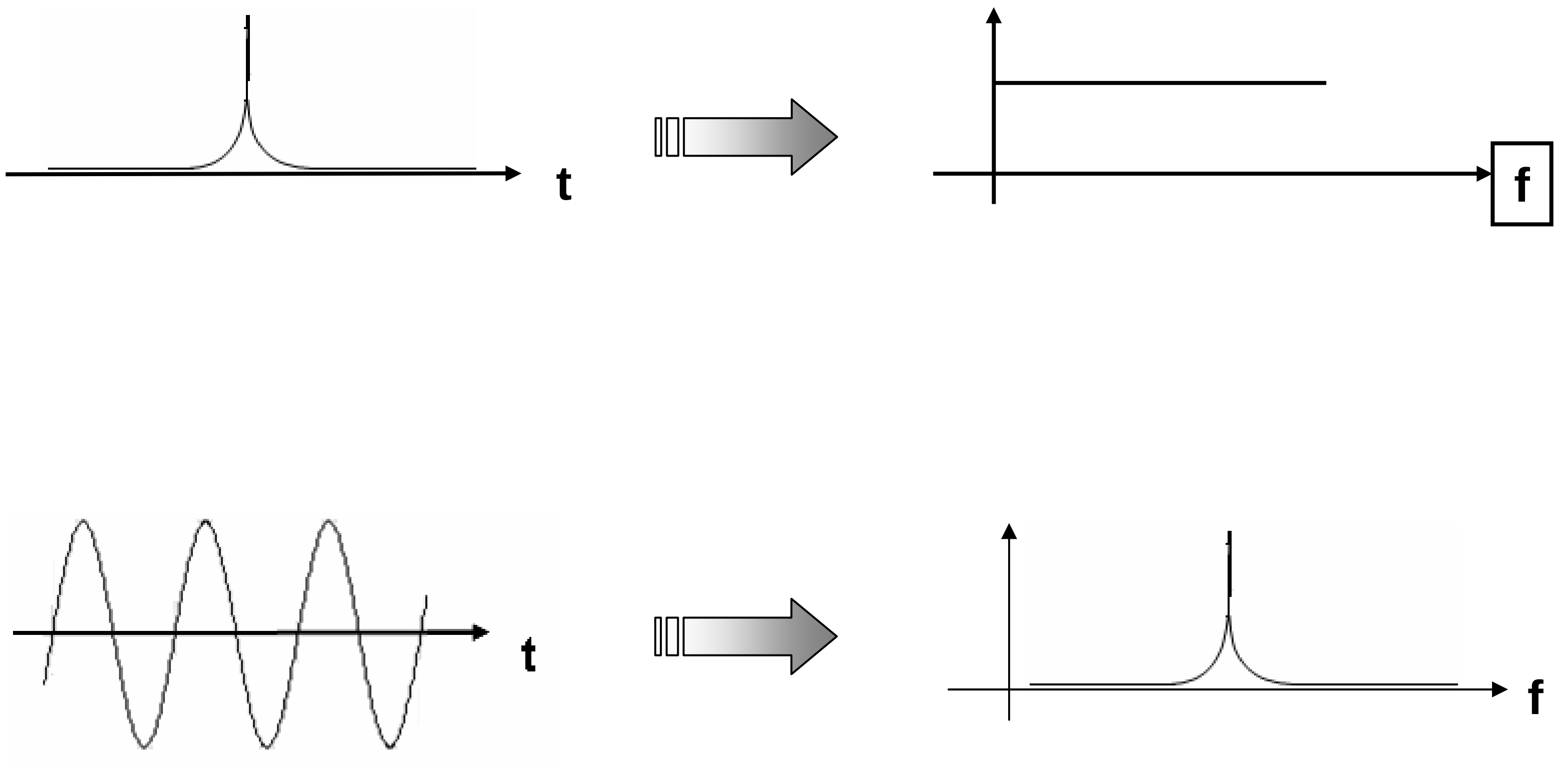
$$f_H = 2 f_L$$

Number of iteration

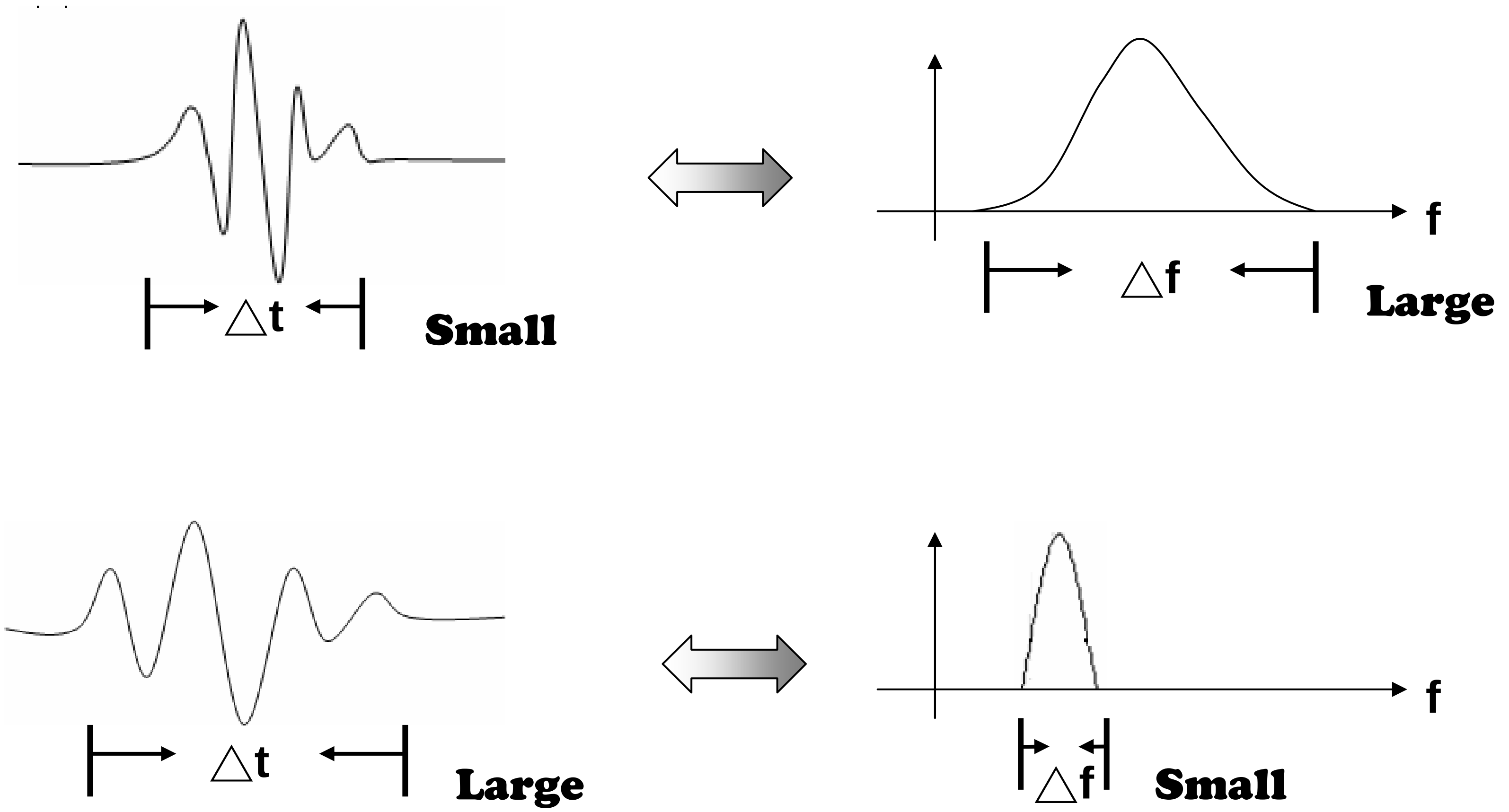
$$m = 33$$

Uncertainty Principle

Simultaneously Determine Time and Frequency



Time and Frequency Resolutions



Formula

$$t_0 = \frac{1}{\|g(t)\|^2} \int_{-\infty}^{\infty} t |g(t)|^2$$

$$\omega_0 = \frac{1}{\|G(\omega)\|^2} \int_{-\infty}^{\infty} \omega \|G(\omega)\|^2 \, d\omega$$

$$\Delta t^2 = \frac{1}{\|g(t)\|^2} \int_{-\infty}^{\infty} (t - t_0)^2 |g(t)|^2 \, dt$$

$$\Delta \omega^2 = \frac{1}{\|G(\omega)\|^2} \int_{-\infty}^{\infty} (\omega - \omega_0)^2 |G(\omega)|^2 \, d\omega$$

$$\Rightarrow \Delta t \Delta \omega \geq \frac{1}{2}$$

Equality holds when

$$g(t) \sim e^{-t^2/4\alpha^2 + j\beta t}$$

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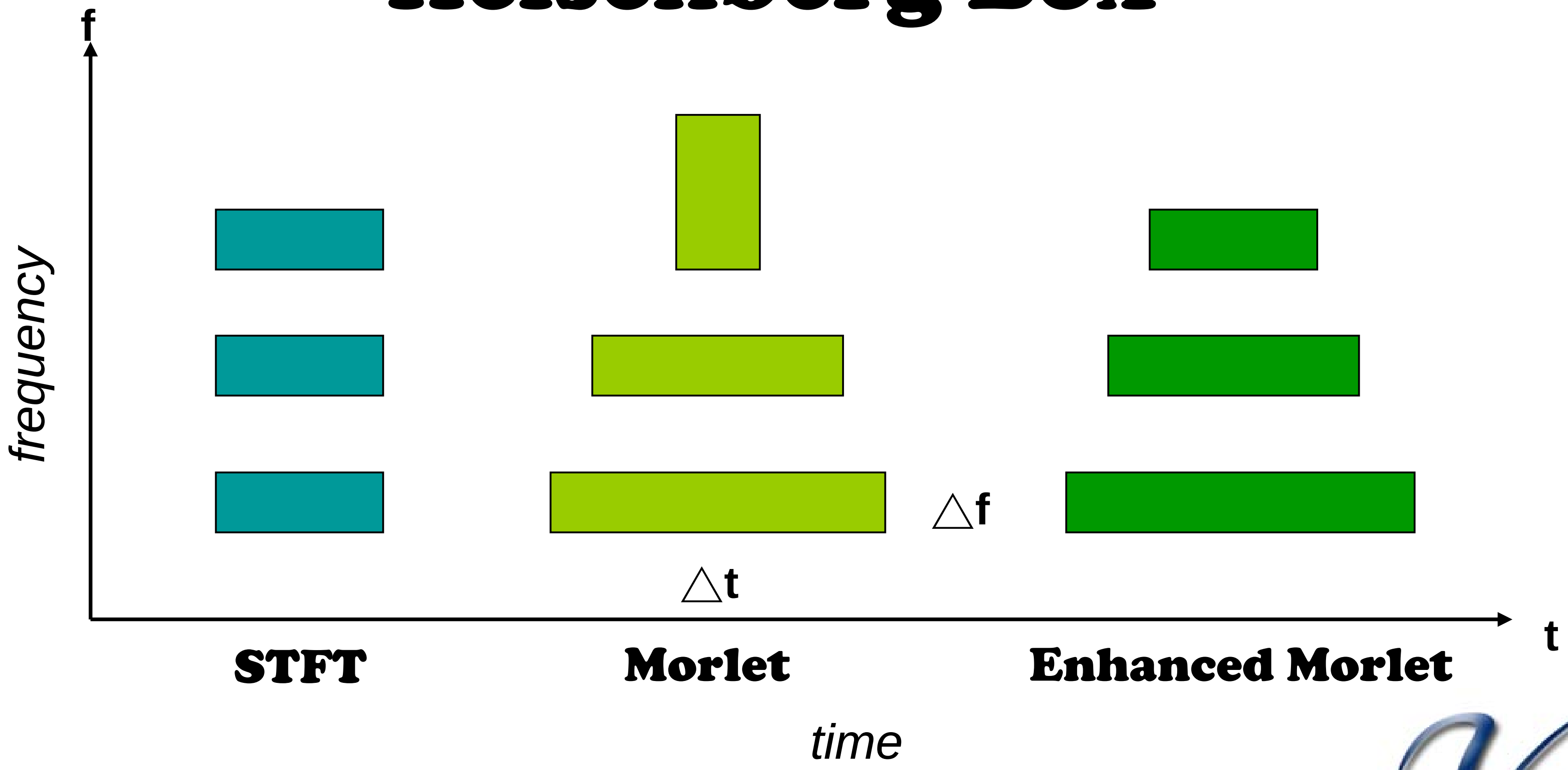
Time-Frequency Resolution

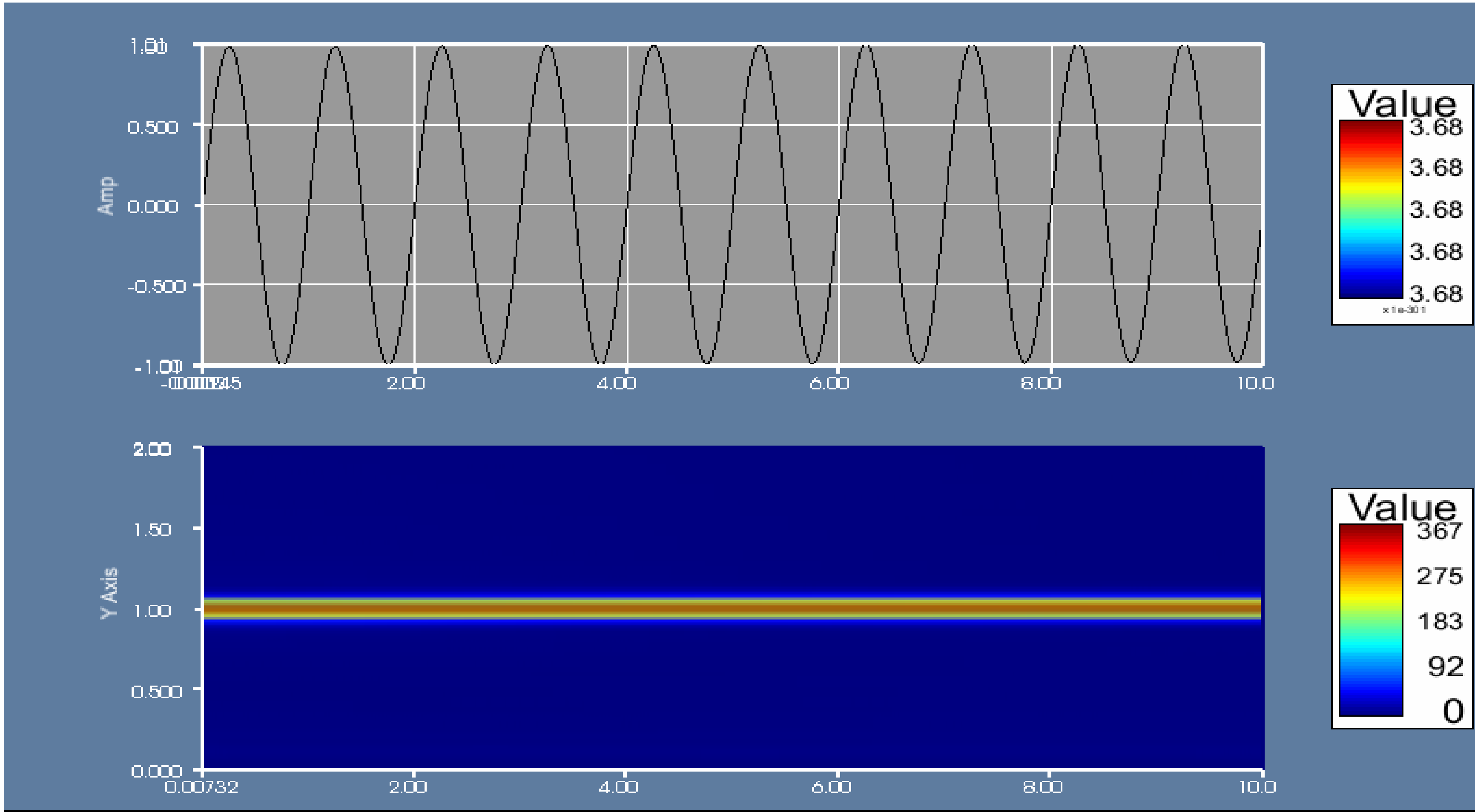
Heisenberg Box



Time-frequency analysis and instantaneous frequency

1. What is time-frequency analysis?
2. The need for instantaneous frequency
3. Methods used for time-frequency analysis

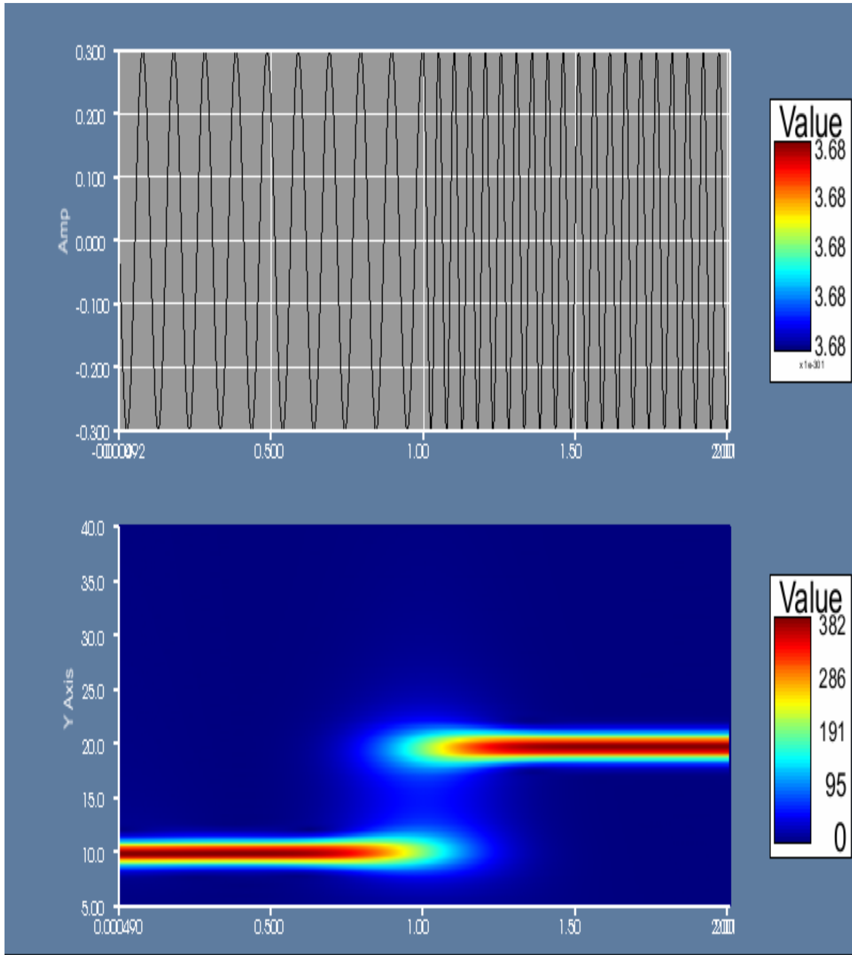
TF Plot: Single frequency



TF Plot: Change of frequency

- Signal with abrupt change of frequency.

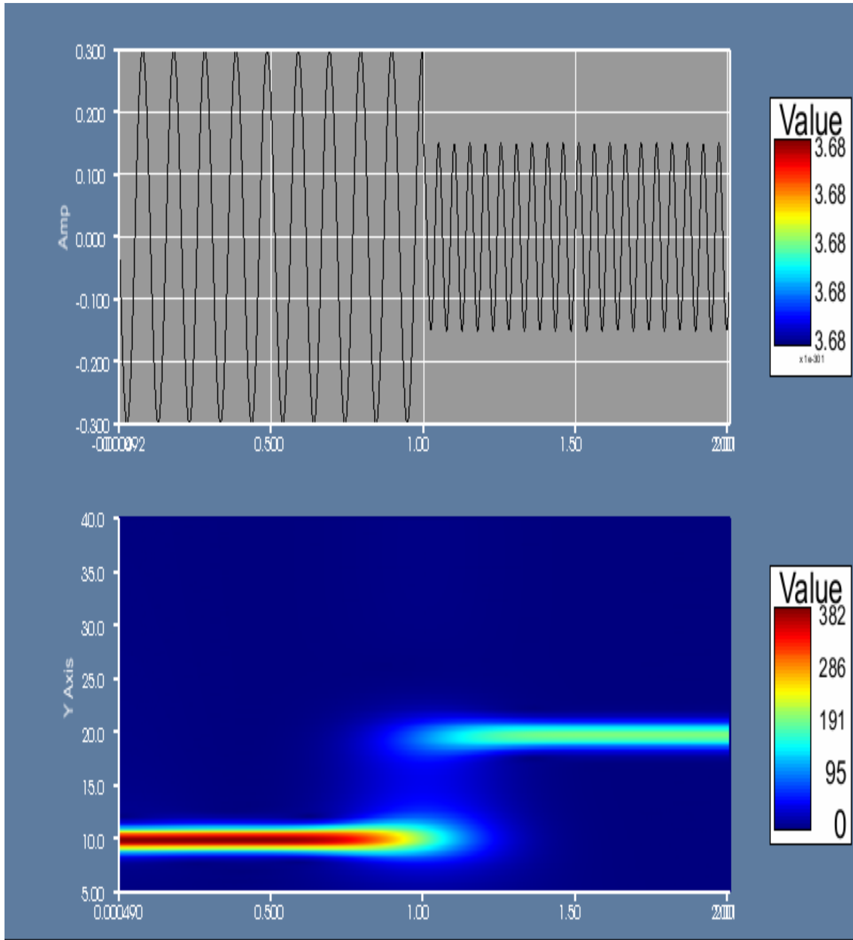
$$x(t) = \begin{cases} 0.30\cos(2\times 10\pi t) & ,0 \le t < 1 \\ 0.30\cos(2\times 20\pi t) & ,1 \le t < 2 \end{cases}$$



TF Plot: Change of frequency and amplitude

- Signal with abrupt change of frequency and amplitude

$$x(t) = \begin{cases} 0.30\cos(2\times 10\pi t) & ,0 \le t < 1 \\ 0.15\cos(2\times 20\pi t) & ,1 \le t < 2 \end{cases}$$



Time-Frequency Analysis Comparison

	Fourier Transform	STFT	Morlet / Enhanced Morlet	Hilbert Transform	HHT
Instantaneous frequency	n/a	distribution	distribution	Single value	Discrete values
Frequency change with time	no	yes	yes	yes	yes
Frequency resolution	good	ok	ok/good	good	good
Adaptive base	no	no	no	n/a	yes
Handling non-linear effect	n/a	no	no	yes	yes

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Short-Term Fourier Transform

- Frequency at a certain time is a distribution obtained from Fourier transform. The short period of signal applied to the Fourier transform contains the specific moment of interest.
- In time-frequency analysis, such idea evolves as the Short-Term Fourier Transform (STFT).

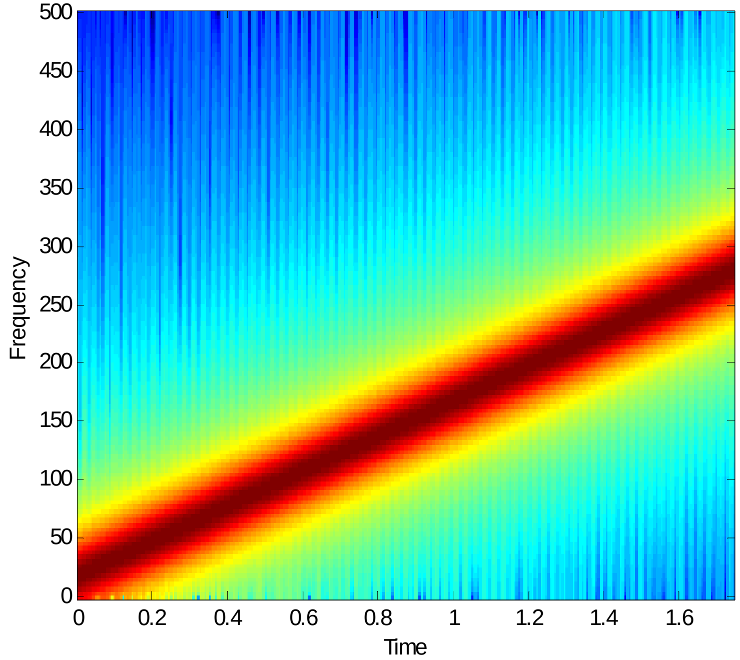
STFT:

$$F(t,\omega) = \int_{-\infty}^{+\infty} f(\tau) g(\tau - t) e^{-i\omega\tau} d\tau$$

Note g is a windowing function.
For Gaussian window, the transform is also known as Gabor Transform.

Gabor transform

- Same chirp signal processed by Gabor transform. (MATLAB)



Challenges in STFT

- Catching low frequency component needs longer time.
- Windowing is needed to avoid end effects.

=> STFT is suitable for band-limited signal like speech and sound.

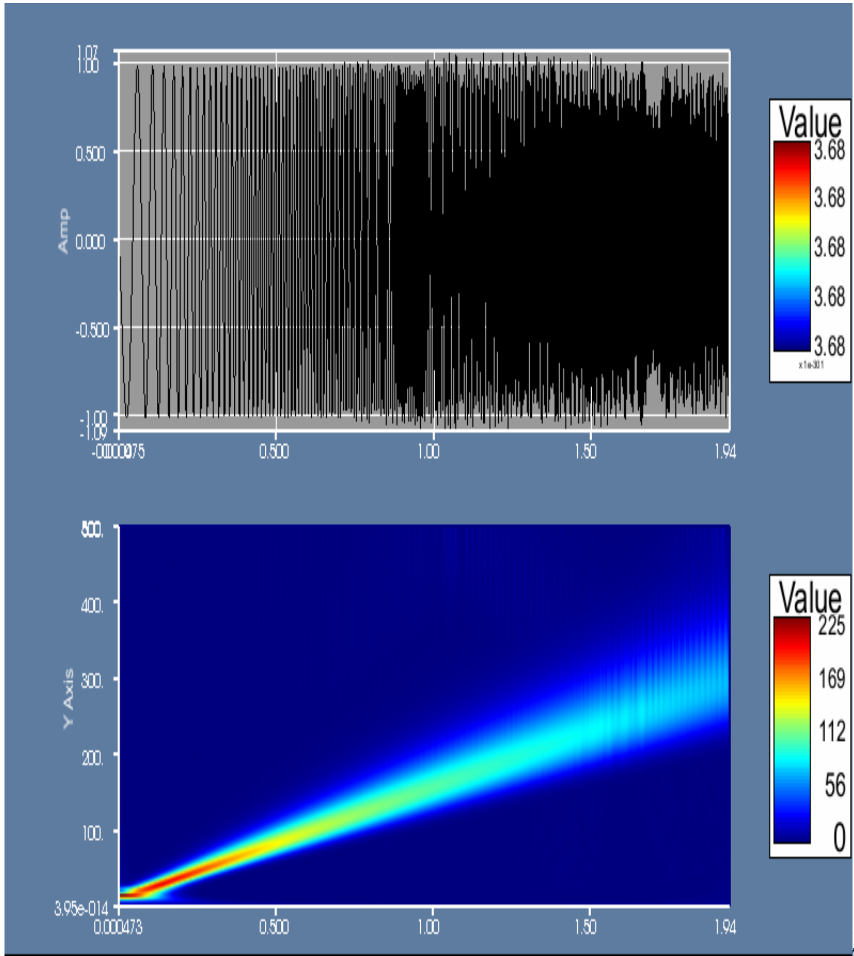
Morlet Transform

- Scale property in signal is related to frequency property when mother wavelet is Morlet..
- Longer duration of wavelet is used to catch lower frequency component.

$$F(t,s)=\int_{-\infty}^{+\infty} f(\tau)\frac{1}{\sqrt{s}}\psi^{*}(\frac{\tau-t}{s})d\tau$$

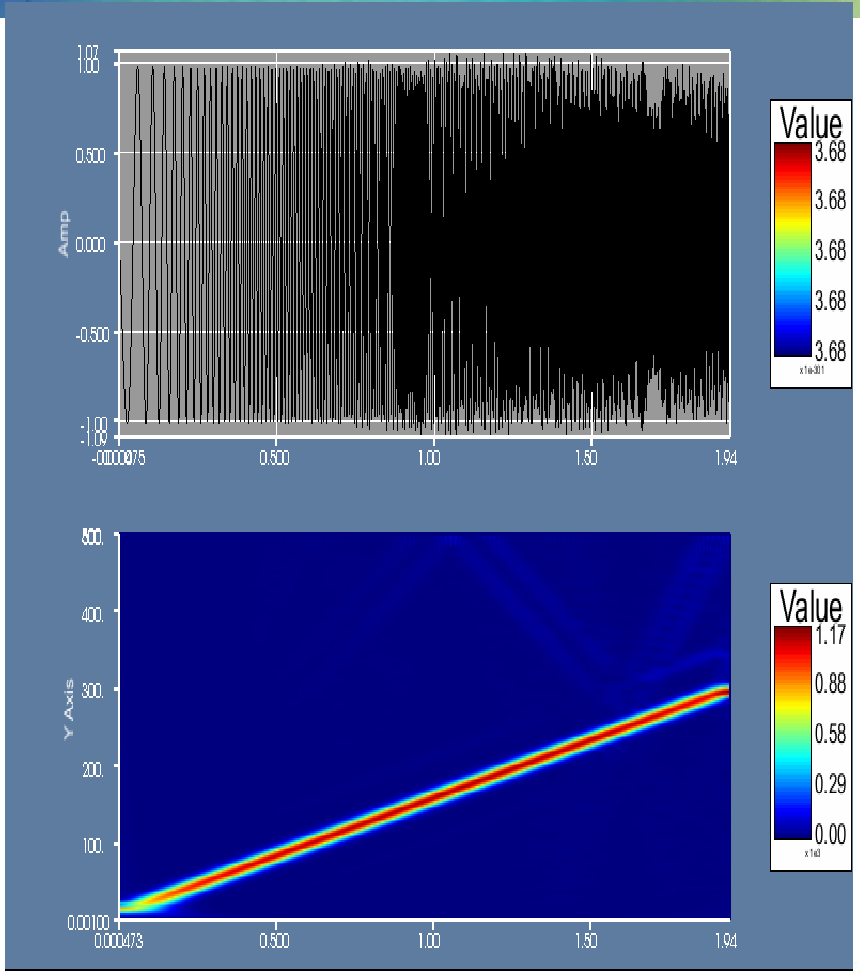
Morlet transform

- Morlet transform on a chirp signal.
- In catching the high frequency spectrum, mother wavelet of short duration of time is used. The spectrum of such wavelet suffers from wide span of frequency, resulting in low resolution, as shown in the right left plot.



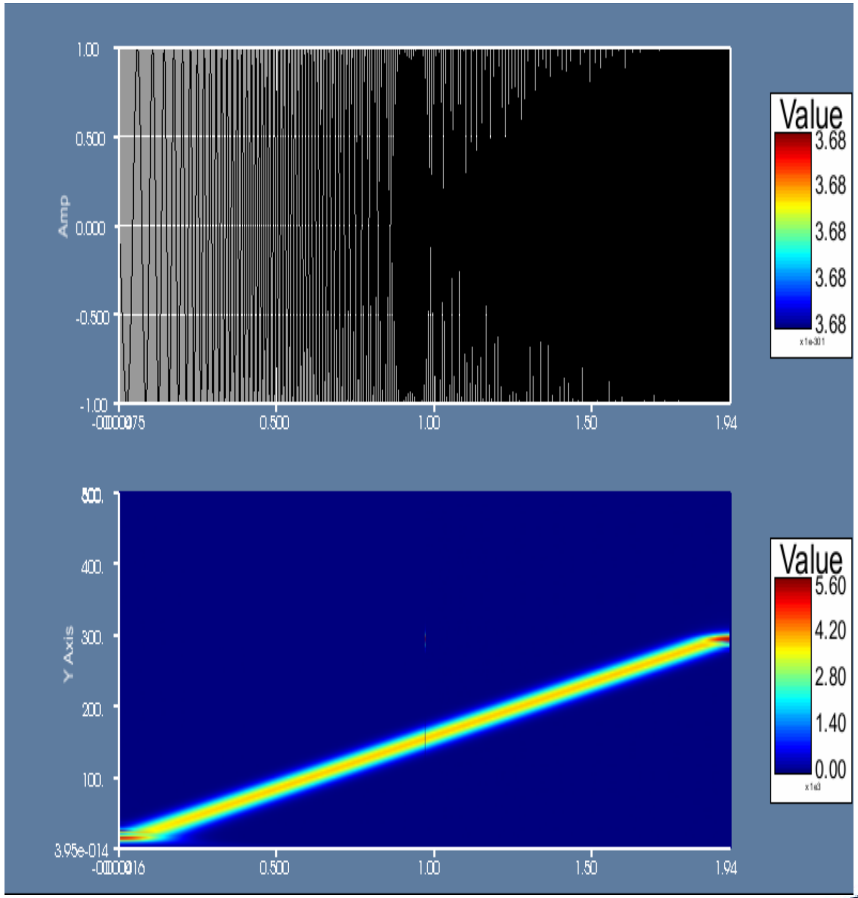
Enhanced Morlet Transform

- By applying Gaussian windowing in frequency domain and knowing that the crossed term of convolution between mother wavelet and signal is the cause of blur, the resolution of Morlet transform can be greatly improved by neglecting the crossed term.
- The fine structure appears in high frequency region is caused by under sampling. The chirp signal is digitized with constant sampling rate.



Chirp signal with higher sampling rate

- With higher sampling rate, the fine structure in TF plot disappeared. This is to assure the fine structure pattern is caused by under sampling.



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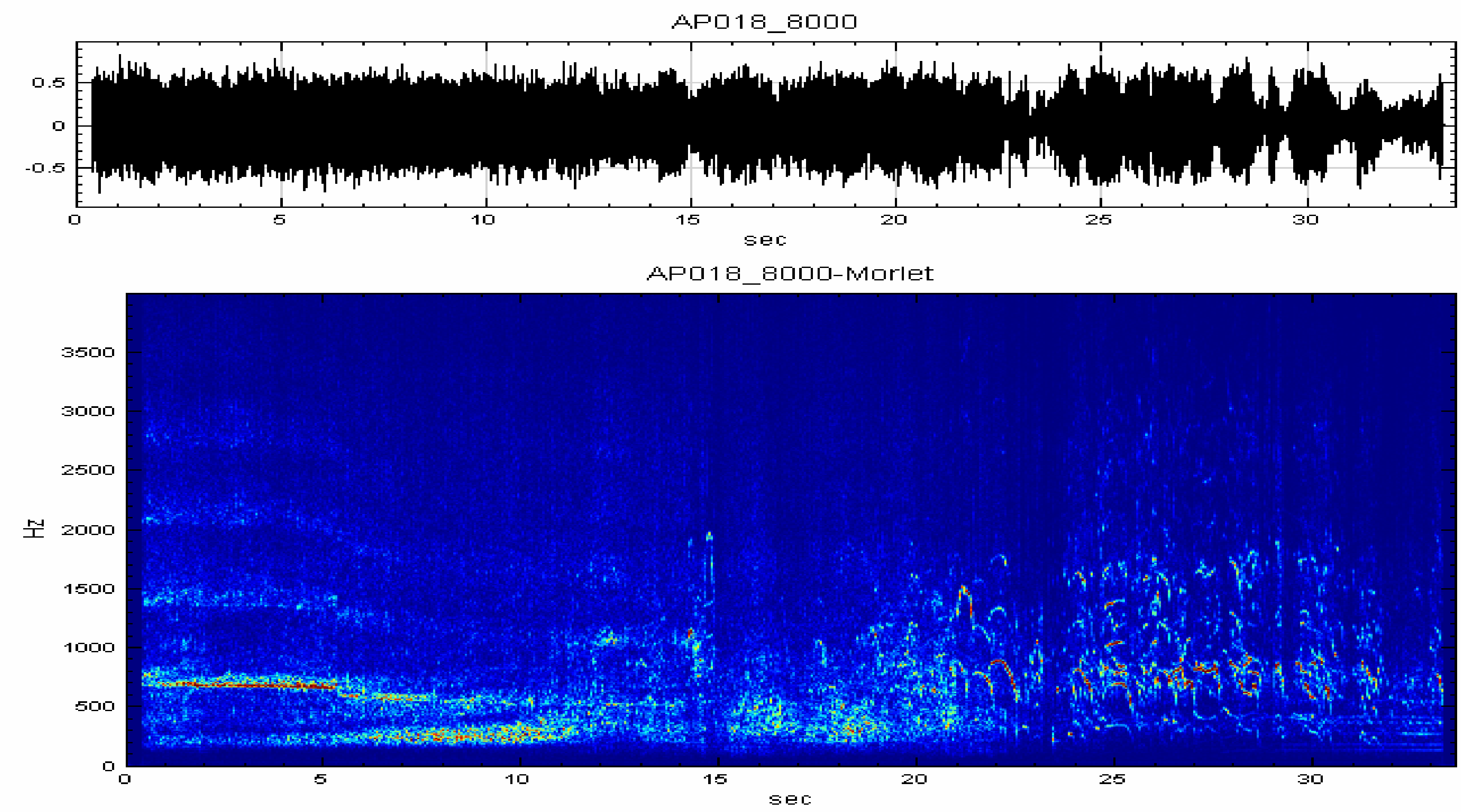

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Empirical Mode Decomposition

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Intrinsic Mode Function (IMF)

- Definition: “Any function having the same numbers of zero-crossings and extrema, and also having symmetric envelopes defined by local maxima and minima respectively is defined as an Intrinsic Mode Function (IMF). “ ~ Norden E. Huang
- An IMF enjoys good properties of Hilbert transform.
- A signal can be regarded as composition of several IMFs. IMF can be obtained through a process called Empirical Mode Decomposition. (ref. Dr. Hsieh’s presentation file)

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Instantaneous Frequency and Hilbert Transform

Prevailing Views of Instantaneous Frequency*

- The term, Instantaneous Frequency, should be banished forever from the dictionary of the communication engineer.
    - » J. Shekel, 1953
- The uncertainty principle makes the concept of an Instantaneous Frequency impossible.
    - » K. Grochennig, 2001

* Transcript of slide from Norden E. Huang

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Definition of Frequency*

- Fourier Analysis
- Wavelet Analysis
- Wigner-Ville Analysis
- Dynamic System through Hamiltonian:

$H(p, q, t)$ and wave action: $A(t) = \int_0^{TC} p\,dq$

$$\omega = \frac{\partial H}{\partial A}$$

- Teager Energy Operator
- Period between zero-crossings and extrema
- HHT analysis: $x(t) = \sum_j a(t) e^{i\theta_j(t)}$ $\omega_j = \frac{d\theta_j}{dt}$

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The need for Instantaneous Frequency

- In classical wave theory, frequency is defined as time rate change of phase.
- Frequency is an continuous concept. There is need to clarify what we mean by frequency at a certain moment.
- Uncertainty principle does apply.

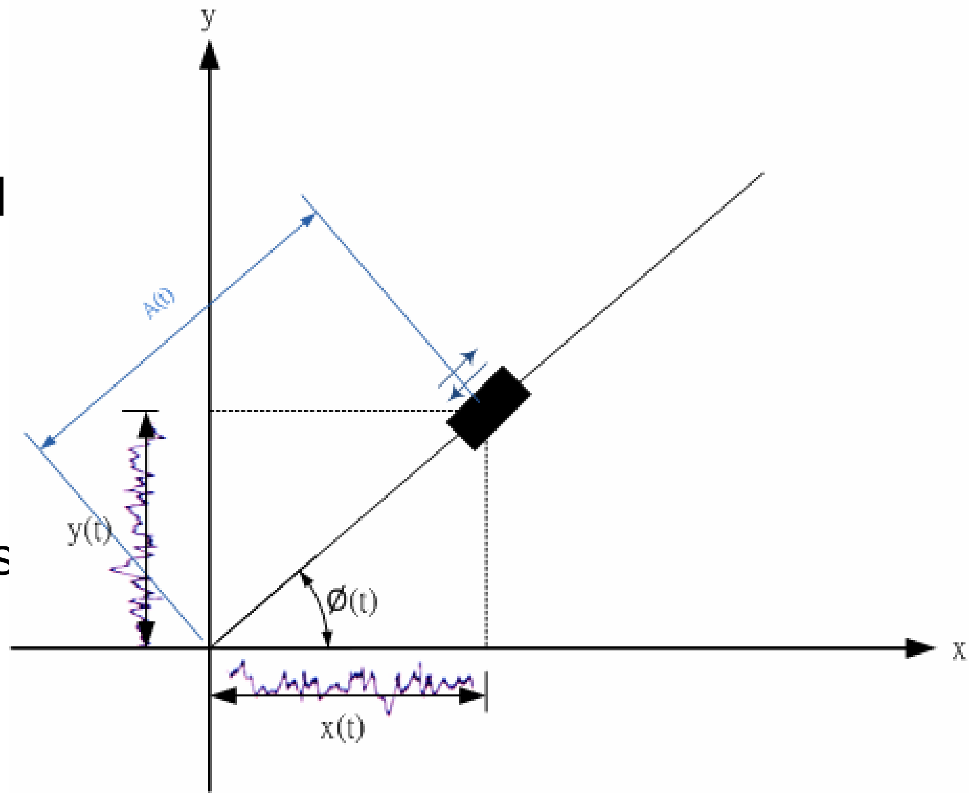
$$k = \nabla \varphi$$

$$\omega = - \frac{\partial \varphi}{\partial t}$$

$$\frac{\partial k}{\partial t} + \nabla \omega = 0$$

Instantaneous Frequency

- A time-series signal can be regarded as the x-axis projection of a planar slider motion. The slider moves along a rotating stick with axial velocity $\dot{a}(t)$, while the stick is spinning with an angular speed $\dot{\varphi}(t)$.
- The corresponding y-axis projection shares the same frequency distribution as x-axis signal with 90 degree phase shift. Such conjugate signal is evaluated from Hilbert Transform.



Hilbert Transform

$$y(t) = \frac{1}{\pi} PV \int \frac{x(\tau)}{t-\tau} d\tau$$

$$z(t) = x(t) + iy(t) = a(t)e^{i\varphi(t)}$$

$$a(t) = \sqrt{x^2 + y^2}$$

$$\varphi(t) = \tan^{-1}\frac{y(t)}{x(t)}$$

$$z(t) = \frac{2}{\sqrt{2\pi}}\int_0^{\infty} S(\varpi)e^{iwt}d\varpi$$

where

$$S(\varpi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x(t)e^{-iwt}dt$$

Instantaneous frequency: $\omega = -\frac{\partial\varphi}{\partial t}$

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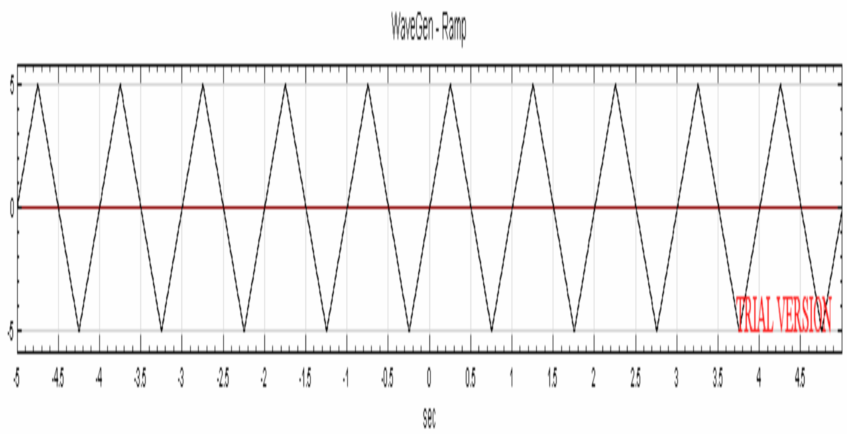
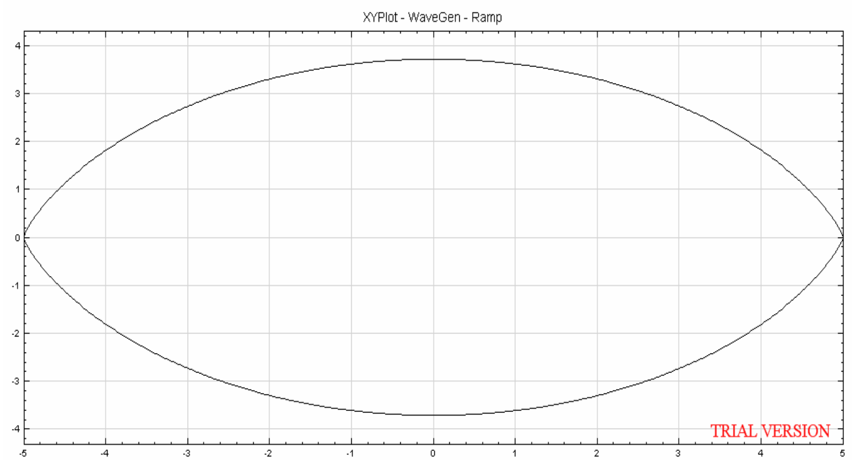
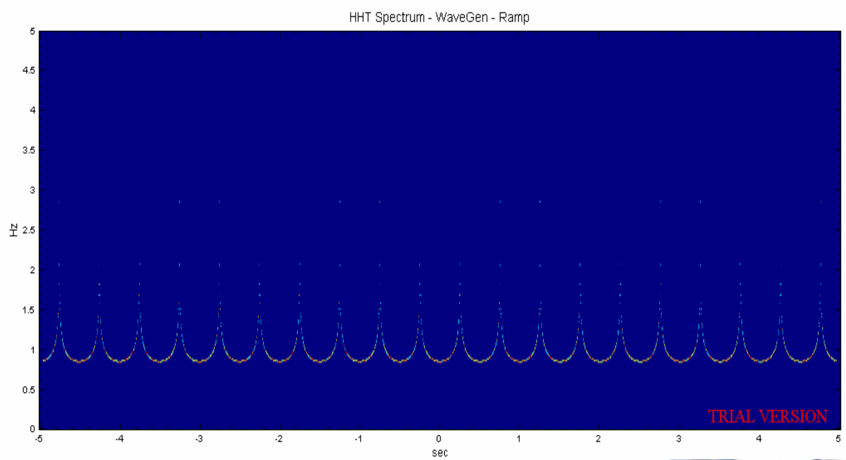
Hilbert transform of triangular signal





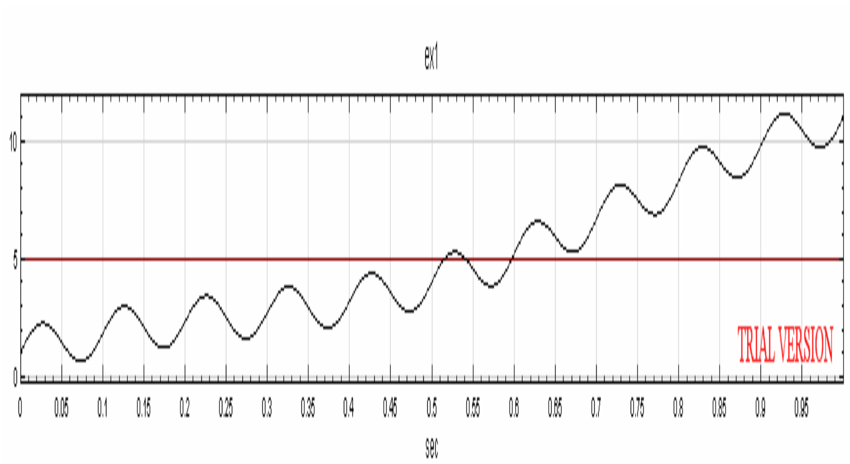
Phase diagram

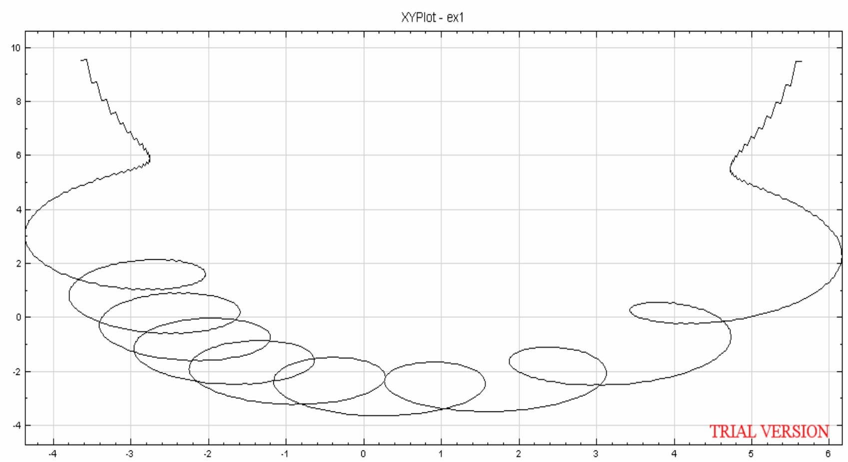


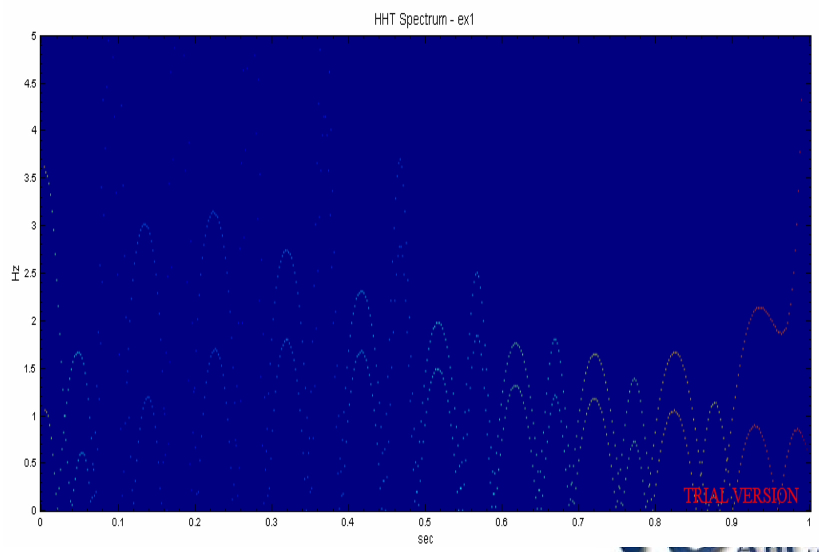
Hilbert spectrum

Drawback of Hilbert Transform

- Negative instantaneous frequency occurs for signal not having equal number of extreme and zero-crossing points.
- Too much DC offset also results in negative frequency.







Limitation on Hilbert Transform*

- Data need to be mono-component. Traditional applications using band-pass filter, which distorts the wave form. (EMD Resolves this problem)
- Bedrosian Theorem: Hilbert transform of [a(t) cos ω(t)] might not be exactly [a(t)sin ω(t)] for arbitrary a and ω. (Normalized HHT resolves this)
- Nuttal Theorem: Hilbert transform of cos ω(t) might not be exactly sin ω(t) for arbitrary ω(t). (Normalized HHT improved on the error bound).
- But we have to realize that the Instantaneous Frequency us always an approximation.

* Transcript of slide from Norden E. Huang

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Limitation for IF computed through Hilbert Transform*

- Data must be expressed in terms of Simple Oscillatory Function (aka Intrinsic Mode Function). (Note: Traditional applications using band-pass filter distorts the wave form; therefore, it can only be used for linear processes.) IMF is only necessary but not sufficient.
- Bedrosian Theorem: Hilbert transform of a(t)cos θ (t) might not be exactly a(t)sin θ (t). Spectra of a(t) and cos θ (t) must be disjoint.
- Nuttal Theorem: Hilbert transform of cos θ (t) might not be sin θ (t) for an arbitrary function of θ (t). Quadrature and Hilbert Transform of arbitrary real functions are not necessarily identical.
- Therefore, simple derivative of the phase of the analytic function might not work.

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Bedrosian Theorem*

- Let f(x) and g(x) denotes generally complex functions in L2(- ∞, ∞) of the real variable x. If
  (1) The Fourier transform F(ω) of f(x) vanished for $|\omega| > a$ and the Fourier transform G(ω) of g(x) vanishes for $|\omega| < a$, where a is an arbitrary positive constant, or
  (2) f(x) and g(x) are analytic (i.e., their real and imaginary parts are Hilbert pairs),

  then the Hilbert transform of the product of f(x) and g(x) is given

  H{ f(x) g(x) } = f(x) H{ g(x) }.

Bedrosian, E. 1963: A Product theorem for Hilbert Transform, Proceedings of IEEE, 51, 868-869.

* Transcript of slide from Norden E. Huang

Nuttal Theorem*

- For any function x(t), having a quadrature xq(t), and a Hilbert transform xh(t); then,

$$E = \int_{0}^{\infty} \left| xq(t) - xh(t) \right|^{2} dt$$

$$= 2 \int_{-\infty}^{0} \left| F_{q}(\omega) \right|^{2} d\omega$$

where Fq(ω) is the spectrum of xq(t).

Nuttal, A. H., 1966: On the quadrature approximation to the Hilbert Transform of modulated signal, Proc. IEEE, 54, 1458

* Transcript of slide from Norden E. Huang

Difficulties with the Existing Limitations*

- We can use EMD to obtain the IMF.
- IMF is only necessary but not sufficient.
- Bedrosian Theorem adds the requirement of not having strong amplitude modulations.
- Nuttal Theorem further points out the difference between analytic function and quadrature. The discrepancy is given in term of the quadrature spectrum, which is an unknown quantity. Therefore, it cannot be evaluated.
- Nuttal Theorem provides a constant limit not a function of time; therefore, it is not very useful for non-stationary processes.

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Instantaneous Frequency: different methods

- Teager Energy Operator
- Normalized IMF
- Generalized Zero-Crossing

Instantaneous Energy Consideration

- Energy of a signal is normally defined as sum of square of amplitude. For signals of different frequencies with same oscillatory amplitude appear to have same “signal energy.”
- In many real situation, energy needed for generating 100Hz signal is much greater than that of 10Hz, even though the amplitudes are the same.
- If signal is generated by physical devices with rotation motion, instantaneously the energy is related to kinetic motion dx(t)/dt and instant work done by the motion.

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Teager Energy Operator

Lie Bracket

$$[x, y] = \dot{x}y - x\dot{y}$$

$$[x, y] / xy = \dot{x} / x - \dot{y} / y$$

If $$y = \dot{x}$$

Energy Operator:

$$\psi(x) = (\dot{x})^2 - x\ddot{x} = [x, \dot{x}]$$

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Discrete energy operator:

$$\psi(x(n)) = x(n)^2 - x(n-1)x(n+1)$$

Instantaneous frequency

$$\Omega = \cos^{-1}(1 - \frac{\psi(x(n) - x(n-1))}{2\psi(x(n))})$$

Amplitude

$$A = (\frac{\psi(x(n))}{1 - (1 - \frac{\psi(x(n) - x(n-1))}{2\psi(x(n))})^2})^{1/2}$$

Comparison of Different Methods*

- TEO extremely local but for linear data only.
- GZC most stable but offers only smoothed frequency over ¼ wave period at most.
- HHT elegant and detailed, but suffers the limitations of Bedrosian and Nuttal Theorems.
- NHHT, with Normalized data, overcomes Bedrosian limitation, offers local, stable and detailed Instantaneous frequency and Error Index for nonlinear and nonstationary data.

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Five Paradoxes on Instantaneous Frequency (a la Leon Cohen)*

- Instantaneous frequency of a signal may not be one of the frequencies in the spectrum.
- For a signal with a line spectrum consisting of only a few sharp frequencies, the instantaneous frequency may be continuous and range over an infinite number of values.
- Although the spectrum of analytic signal is zero for negative frequency, the instantaneous frequency may be negative.
- For the band limited signal the instantaneous frequency may be outside the band.
- The value of the Instantaneous frequency should only depend on the present time, but the analytic signal, from which the instantaneous frequency is computed, depends on the signal values for the whole time space.

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Observations*

- Paradoxes 1, 2 and 4 are essentially the same: Instantaneous Frequency values may be different from the frequency in the spectrum.
- The negative frequency in analytical signal seems to violate Gabor's construction.
- The analytic function, or the Hilbert Transform, involves the functional values over the whole time domain.

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The advantages of using HHT

- Removal of non-periodical part
- Separation of carrier frequency: even though the spectrum is close. Such function can be hardly achieved by frequency based filter.
- Nonlinear effect might introduce frequency harmonics in spectrum domain. Through HHT, the nonlinear effect can be caught by EMD/IMF. The marginal frequency therefore enjoys shorter band width.
- Average frequency in each IMF represents intrinsic signature of physics behind the data.
- Signal can be regarded as generated from rotors of different rotating speeds (analytical signals).

Comparison*

	Fourier	Wavelet	Hilbert
Basis	a priori	a priori	Adaptive
Frequency	Convolution: Global	Convolution: Regional	Differentiation: Local
Presentation	Energy-frequency	Energy-time-frequency	Energy-time-frequency
Nonlinear	no	no	yes
Non-stationary	no	yes	yes
Feature extraction	no	Discrete: no Continous: yes	yes

* Transcript of slide from Norden E. Huang

Outstanding Mathematical Problem*

- Adaptive data analysis methodology in general
- Nonlinear system identification methods
- Prediction problem for nonstationary processes (end effects)
- Optimization problem (the best IMF selection and uniqueness. Is there a unique solution?)
- Spline problem (best spline implement of HHT, convergence and 2-D)
- Approximation problem (Hilbert transform and quadrature)

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Summary*

- The so called paradoxes are really not problems, once some misconceptions are clarified.
- Instantaneous Frequency (IF) has very different meaning than the Fourier frequency.
- IF for special mono-component functions only: IMFs.
- Even for IMFs, there are still problems associated with Hilbert Transform. We have to use Normalized HHT to compute the IF.
- IF is always an approximation.

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Thank you!

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