時 頻 分 析 近 年 來 的 發 展

Recent Development of Time-Frequency Analysis

丁 建 均

國立台灣大學電信工程學研究所

一、什麼是時頻分析 (Time-Frequency Analysis)

Frequency Analysis: by Fourier transform (FT)

$$X(f)=\int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}dt$$

Fourier transform 不足的地方：

無法看出頻率隨著時間而改變的情形

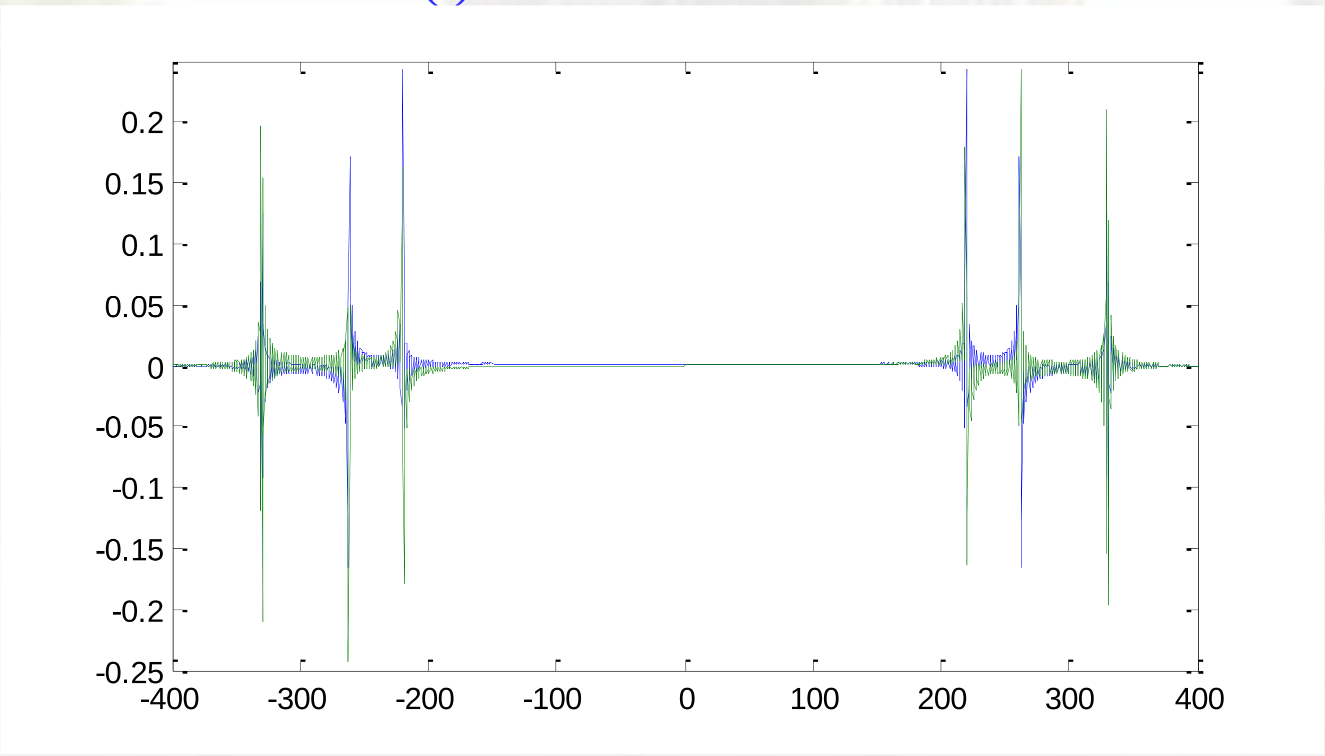
Example 1

$x(t) = \cos(440\pi t)$ when $t < 0.5$, La

$x(t) = \cos(660\pi t)$ when $0.5 \le t < 1$, Me

$x(t) = \cos(524\pi t)$ when $t \ge 1$ Do

The Fourier transform of $x(t)$



Short-Time Fourier Transform

$$X(t, f) = \int_{-\infty}^{\infty} w(t-\tau)\, x(\tau)\, e^{-j2\pi f \tau} d\tau$$

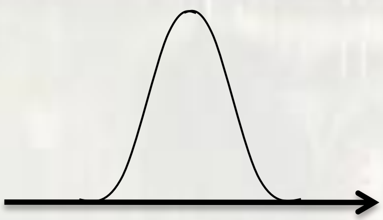
$w(t)$: mask function

也稱作 windowed Fourier transform 或

time-dependent Fourier transform

例如：

$$w(t) = \exp\left(-\sigma t^2\right)$$

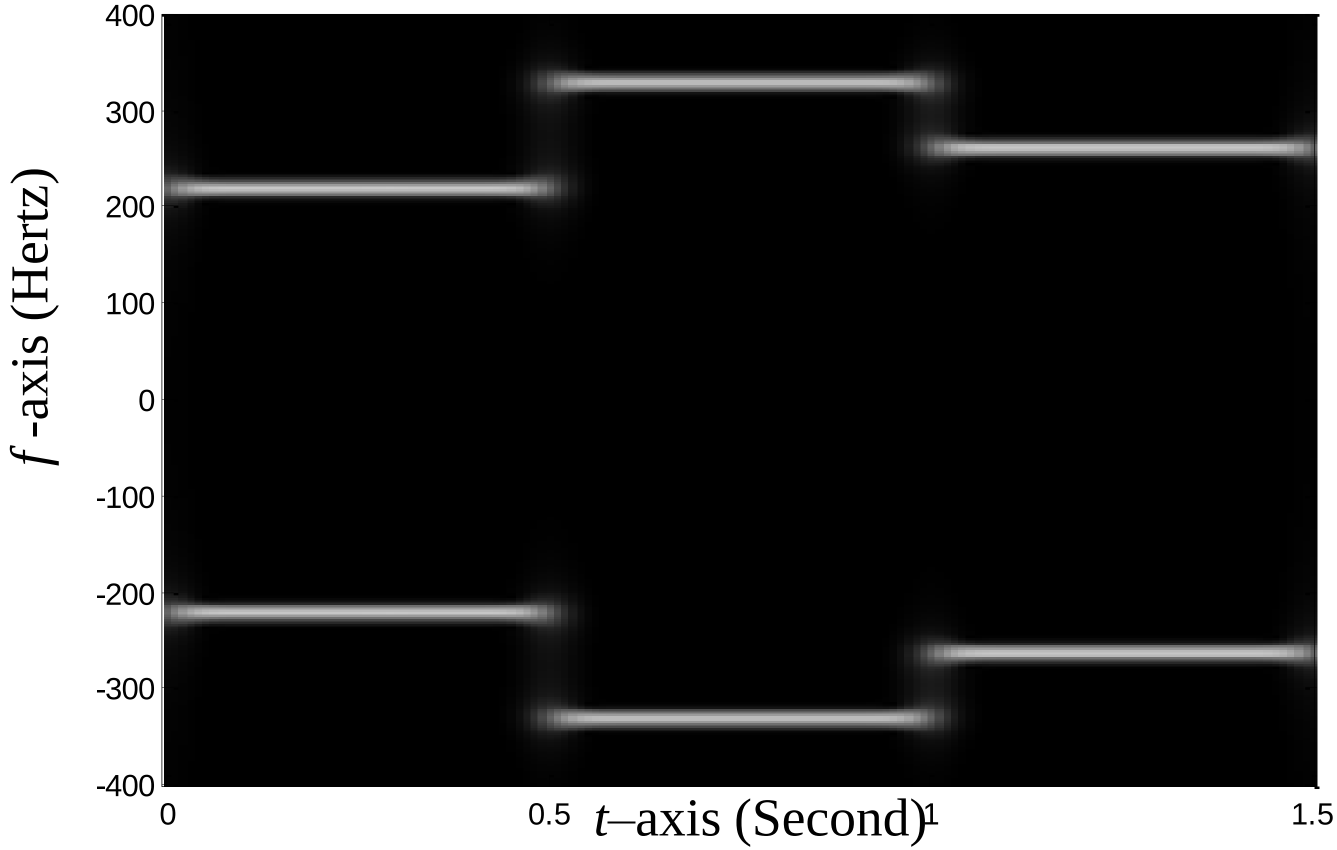


(此時 short-time Fourier transform 被稱作 Gabor transform)

Example: $x(t) = \cos(440\pi t)$ when $t < 0.5$,

$x(t) = \cos(660\pi t)$ when $0.5 \le t < 1$,

$x(t) = \cos(524\pi t)$ when $t \ge 1$



用 Gray level 來表示 $X(t, f)$ 的 amplitude

瞬時頻率 (Instantaneous Frequency)

If $x(t)=\sum_{k=1}^{N}a_k\cdot\exp(j\cdot\phi_k(t))$

then the instantaneous frequency is

$$\frac{\phi_1'(t_0)}{2\pi},\frac{\phi_2'(t_0)}{2\pi},\frac{\phi_3'(t_0)}{2\pi},\cdots,\frac{\phi_N'(t_0)}{2\pi}$$

If the order of $\phi_k(t)$ > 1, then instantaneous frequency varies with time

Example 2

(a) $x(t) = 0.5\cos\left(6400\pi t - 600\pi t^2\right)$ $t \in [0, 3]$

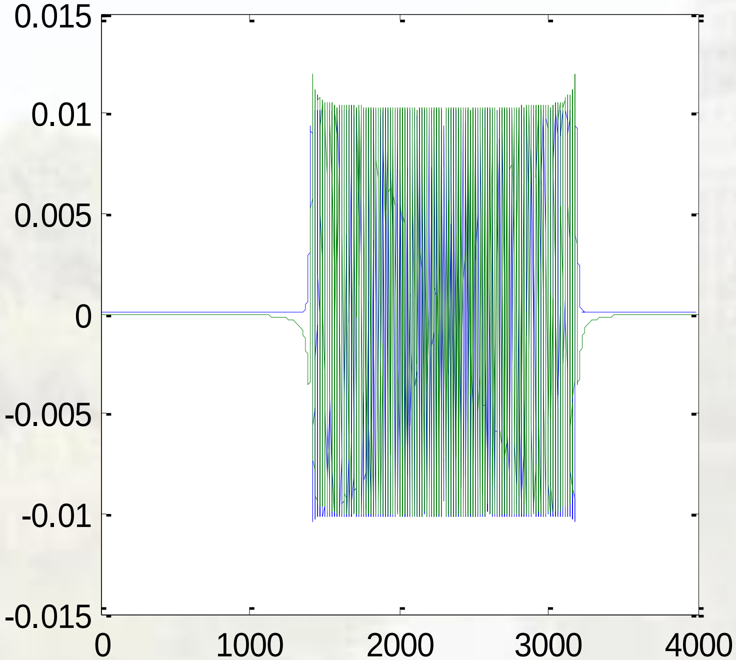
瞬 時 頻 率 $3200 - 600t$

(b) $x(t) = 0.5\cos\left(600\pi t^3 - 2700\pi t^2 + 5050\pi t\right)$ $t \in [0, 3]$

瞬 時 頻 率 $900t^2 - 2700t + 2525$

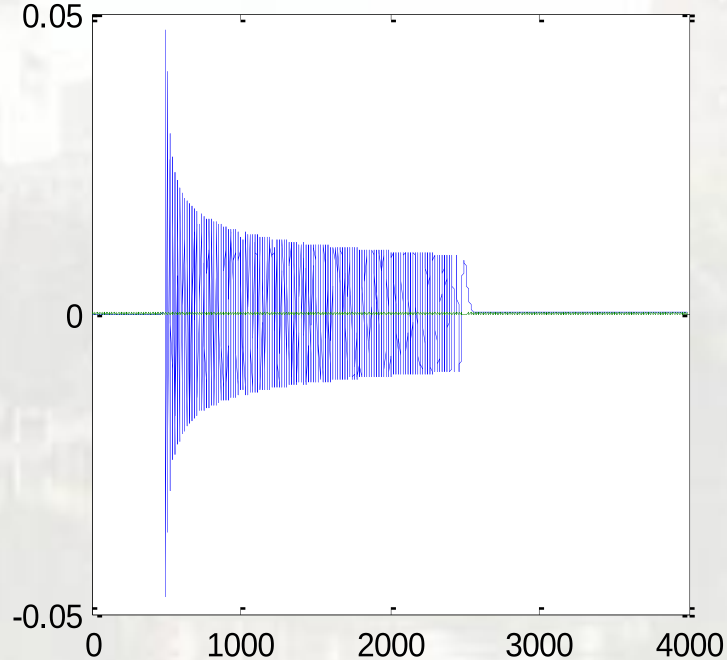
Fourier transform

$$x(t) = 0.5\cos\left(6400\pi t - 600\pi t^2\right)$$



f (Hz)

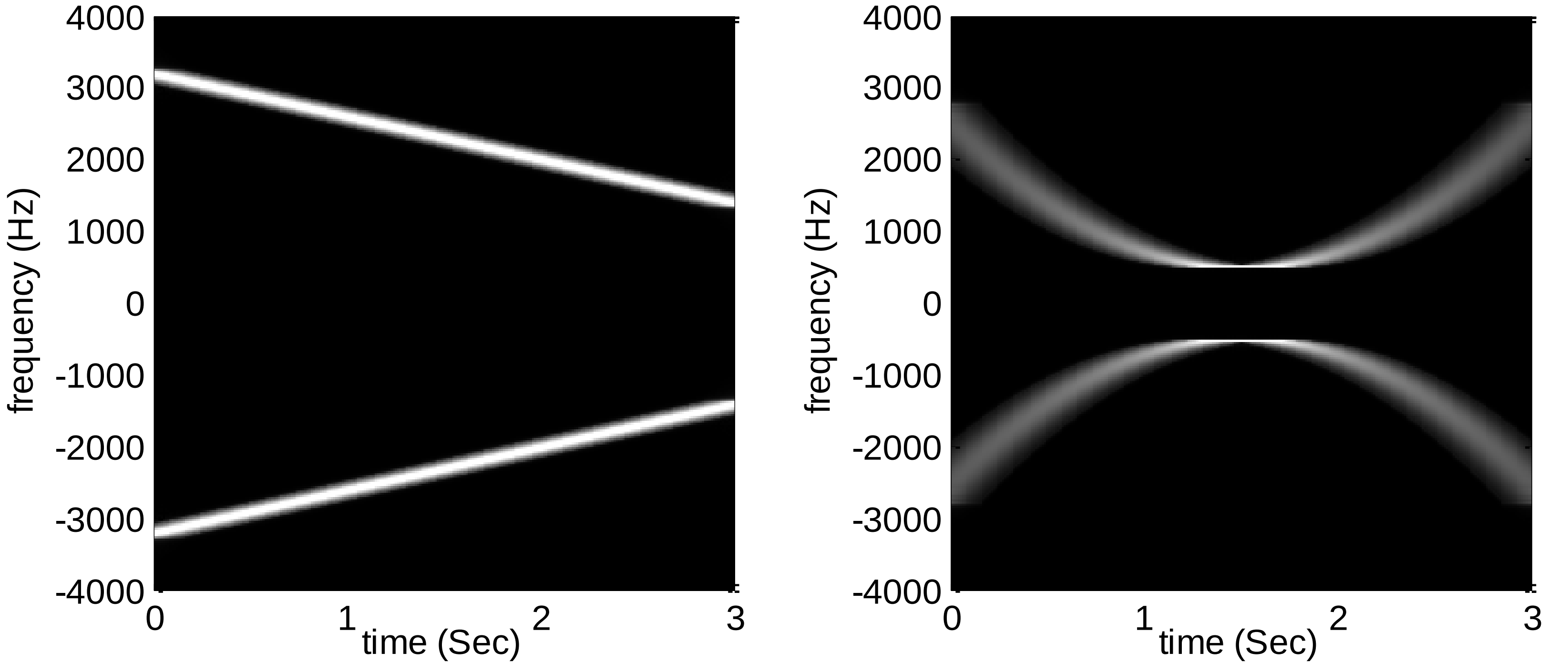
$$x(t) = 0.5\cos\left(600\pi t^3 - 2700\pi t^2 + 5050\pi t\right)$$



f (Hz)

Short-time Fourier transform

$$x(t) = 0.5\cos\left(6400\pi t - 600\pi t^2\right) \qquad x(t) = 0.5\cos\left(600\pi t^3 - 2700\pi t^2 + 5050\pi t\right)$$



頻率會隨著時間而變化的例子：

Frequency Modulation (FM) Signal

Speech

Music

Others (Animal voice, Doppler effect, seismic waves, radar system, optics, rectangular function)

In fact, in addition to **sinusoid-like functions**, the instantaneous frequencies of other functions will inevitably vary with time.

二、時頻分析的分類和發展歷史

時頻分析理論發展年表

AD 1785 The Laplace transform was invented

AD 1812 The Fourier transform was invented

AD 1822 The work of the Fourier transform was published

AD 1910 The Haar Transform was proposed

AD 1927 Heisenberg discovered the uncertainty principle

AD 1929 The fractional Fourier transform was invented by Wiener

AD 1932 The Wigner distribution function was proposed

AD 1946 The short-time Fourier transform and the Gabor transform was proposed. (In the same year, the computer was invented)

AD 1961 Slepian and Pollak found the prolate spheroidal wave function

AD 1966 Cohen's class distribution was invented

AD 1971 Moshinsky and Quesne proposed the linear canonical transform

AD 1980 The fractional Fourier transform was re-invented by Namias

AD 1981 Morlet proposed the wavelet transform

AD 1982 The relations between the random process and the Wigner distribution function was found by Martin and Flandrin

AD 1988 Mallat and Meyer proposed the multiresolution structure of the wavelet transform; In the same year, Daubechies proposed the compact support orthogonal wavelet

AD 1989 The Choi-Williams distribution was proposed; In the same year, Mallat proposed the fast wavelet transform

AD 1990 The cone-Shape distribution was proposed by Zhao, Atlas, and Marks

AD 1993 Mallat and Zhang proposed the matching pursuit; In the same year, the rotation relation between the WDF and the fractional Fourier transform was found by Lohmann

AD 1994 The applications of the fractional Fourier transform in signal processing were found by Almeida, Ozaktas, Wolf, Lohmann, and Pei

AD 1995 L. J. Stankovic, S. Stankovic, and Fakultet proposed the pseudo Wigner distribution

AD 1996 Stockwell, Mansinha, and Lowe proposed the S transform

AD 1998 N. E. Huang proposed the Hilbert-Huang transform

AD 1999 Candes, Donoho, Antoine, Murenzi, and Vandergheynst proposed the directional wavelet transform

AD 2000 The standard of JPEG 2000 was published by ISO

AD 2002 Stankovic proposed the time frequency distribution with complex arguments

AD 2003 Pinnegar and Mansinha proposed the general form of the S transform

AD 2007 The Gabor-Wigner transform was proposed by Pei and Ding

時頻分析理論的五大家族

(1) Short-Time Fourier transform 家族

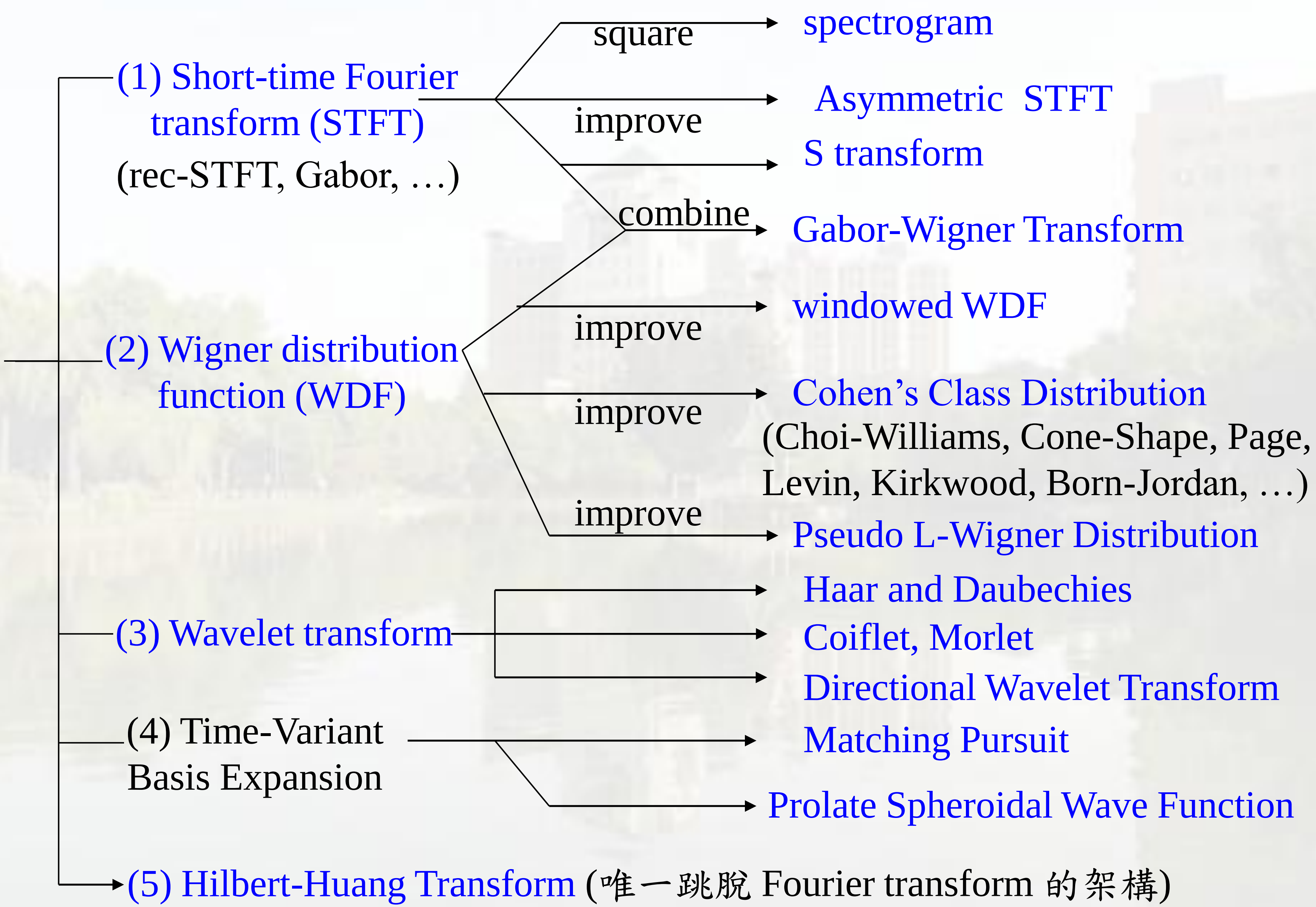
(2) Wigner distribution function 家族

(3) Wavelet transform 家族

(4) Time-Variant Basis Expansion 家族

(5) Hilbert-Huang transform 家族

時頻分析的大家族


(1) Short-time Fourier transform (STFT)
(rec-STFT, Gabor, …)
square
spectrogram
improve
Asymmetric STFT
S transform
combine
Gabor-Wigner Transform
(2) Wigner distribution function (WDF)
improve
windowed WDF
improve
Cohen's Class Distribution
(Choi-Williams, Cone-Shape, Page, Levin, Kirkwood, Born-Jordan, …)
improve
Pseudo L-Wigner Distribution
(3) Wavelet transform
Haar and Daubechies
Coiflet, Morlet
Directional Wavelet Transform
(4) Time-Variant Basis Expansion
Matching Pursuit
Prolate Spheroidal Wave Function
(5) Hilbert-Huang Transform (唯一跳脫 Fourier transform 的架構)

(1) Short-Time Fourier Transform (1946)

$$X(t, f) = \int_{-\infty}^{\infty} w(t-\tau)\, x(\tau)\, e^{-j2\pi f \tau} d\tau$$

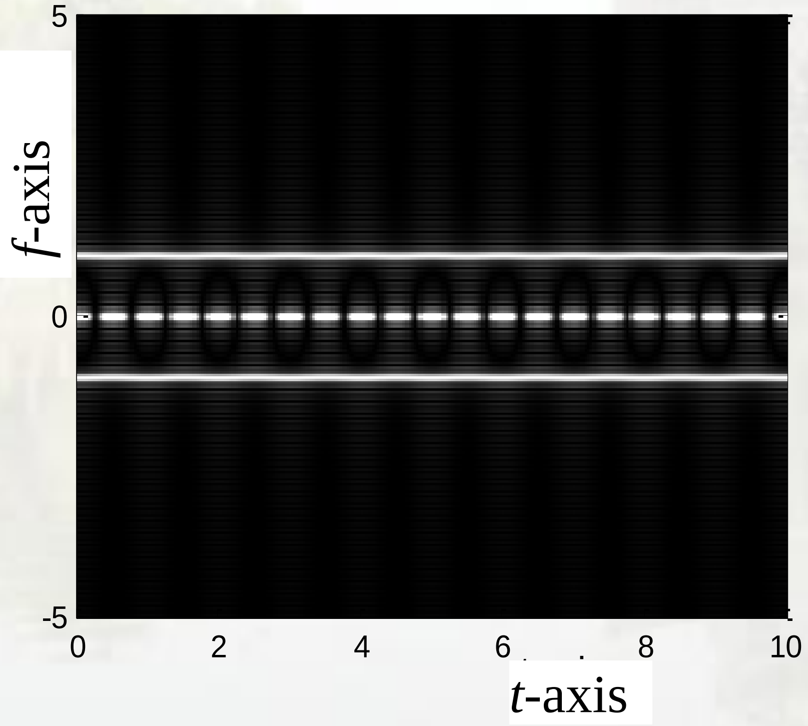
(2) Wigner Distribution Function (1932)

$$W_x(t, f) = \int_{-\infty}^{\infty} x(t+\tau/2) \cdot x^*(t-\tau/2)\, e^{-j2\pi \tau f} \cdot d\tau$$

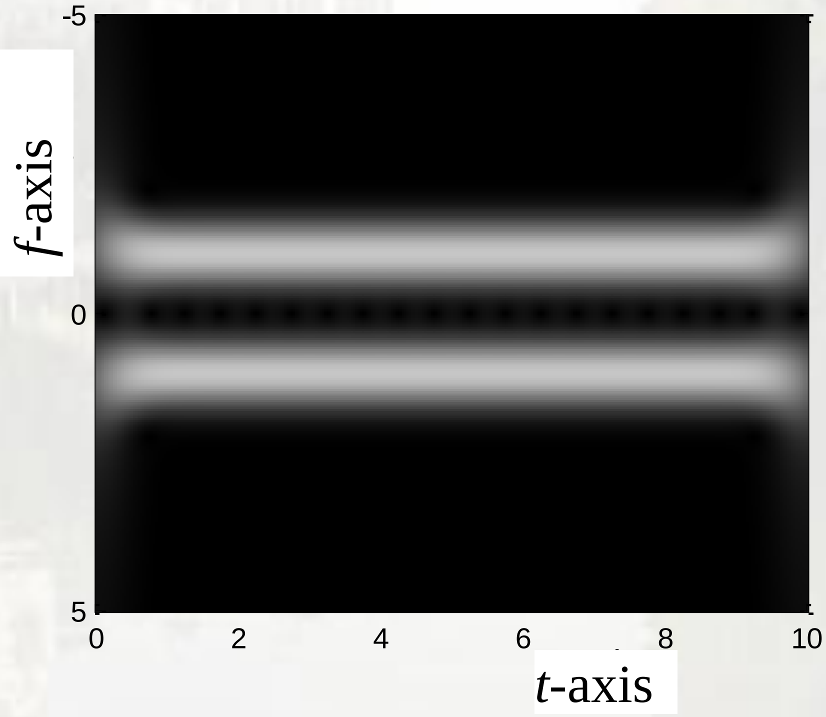
Simulations

$$x(t) = \cos(2\pi t)$$

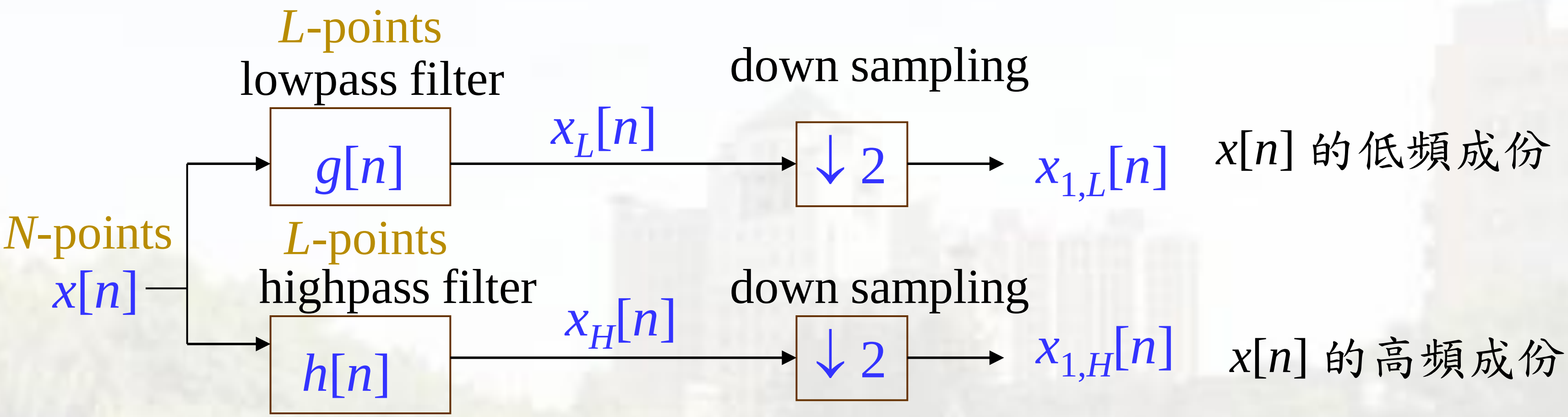
by WDF



by short-time Fourier transform



(3) Wavelet Transform (1981)



$$x_L[n] = \sum_k x[n-k]g[k] \qquad x_{1,L}[n] = \sum_k x[2n-k]g[k]$$

$$x_H[n] = \sum_k x[n-k]h[k] \qquad x_{1,H}[n] = \sum_k x[2n-k]h[k]$$

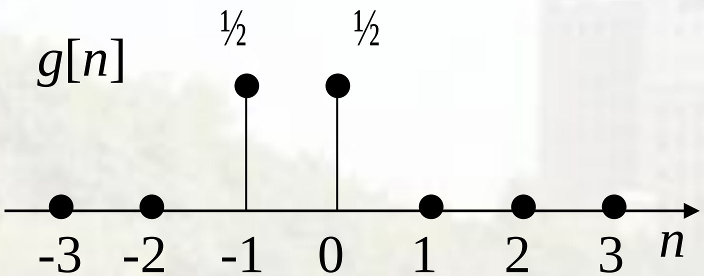
例子：2-point Haar wavelet

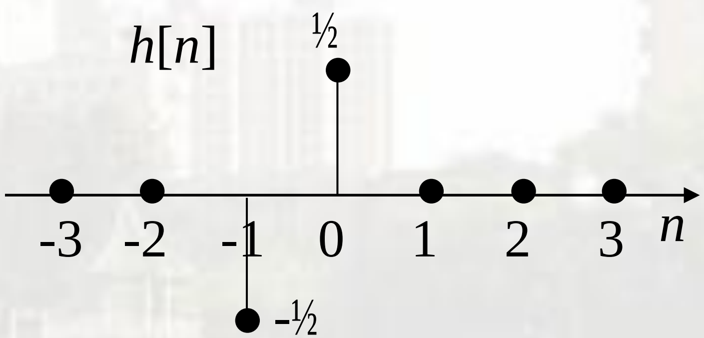
$g[n] = 1/2$ for $n = -1, 0$

$g[n] = 0$ otherwise

$h[0] = 1/2, \quad h[-1] = -1/2,$

$h[n] = 0$ otherwise





then

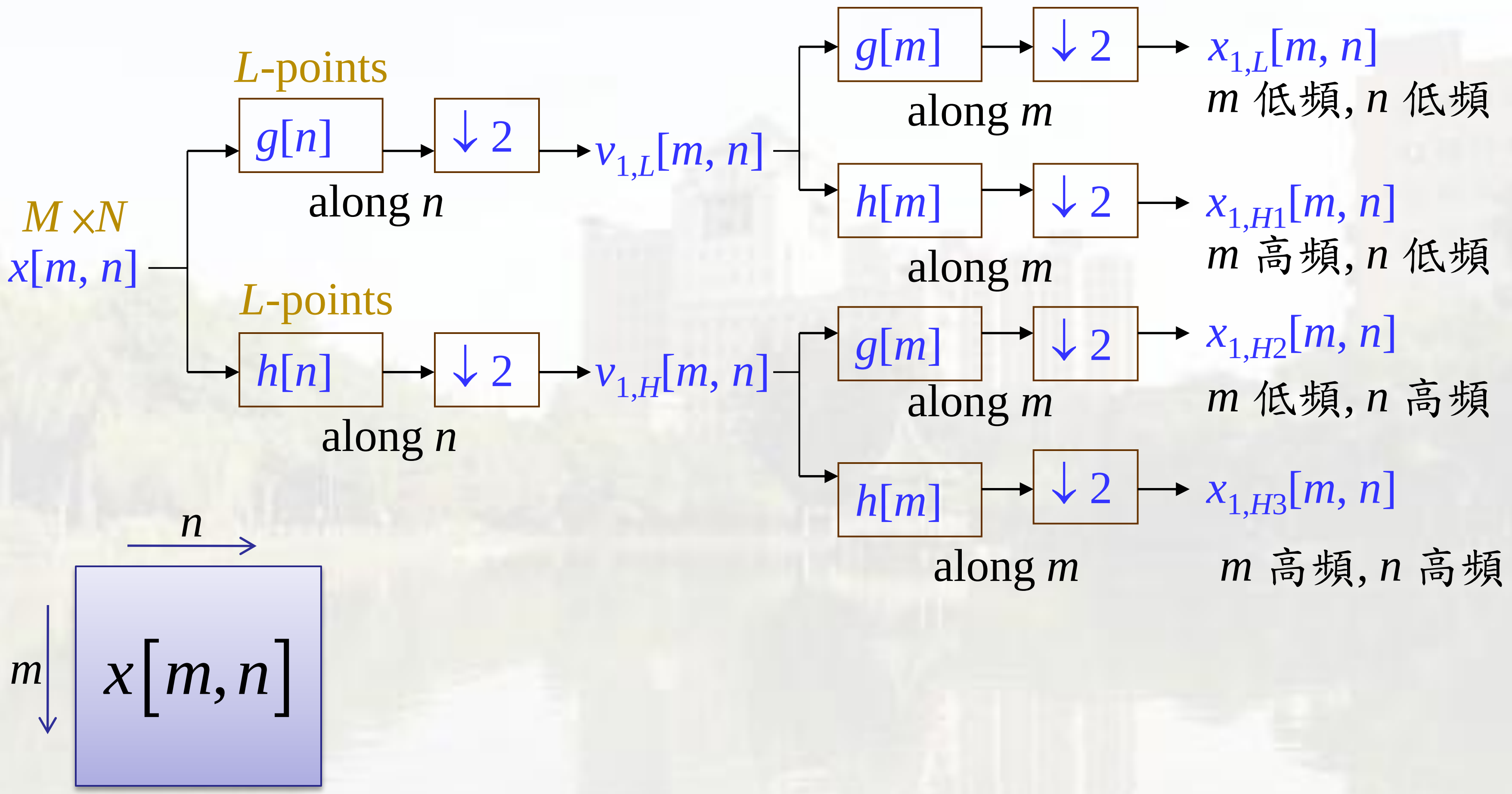
$$x_{1,L}[n] = \frac{x[2n] + x[2n+1]}{2}$$

（兩點平均）

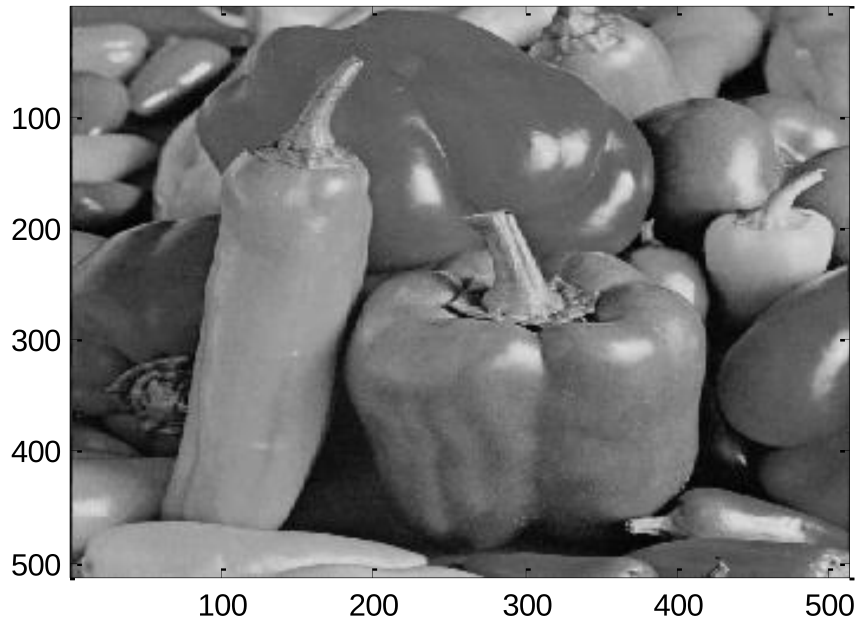
$$x_{1,H}[n] = \frac{x[2n] - x[2n+1]}{2}$$

（兩點之差）

2-D 的情形

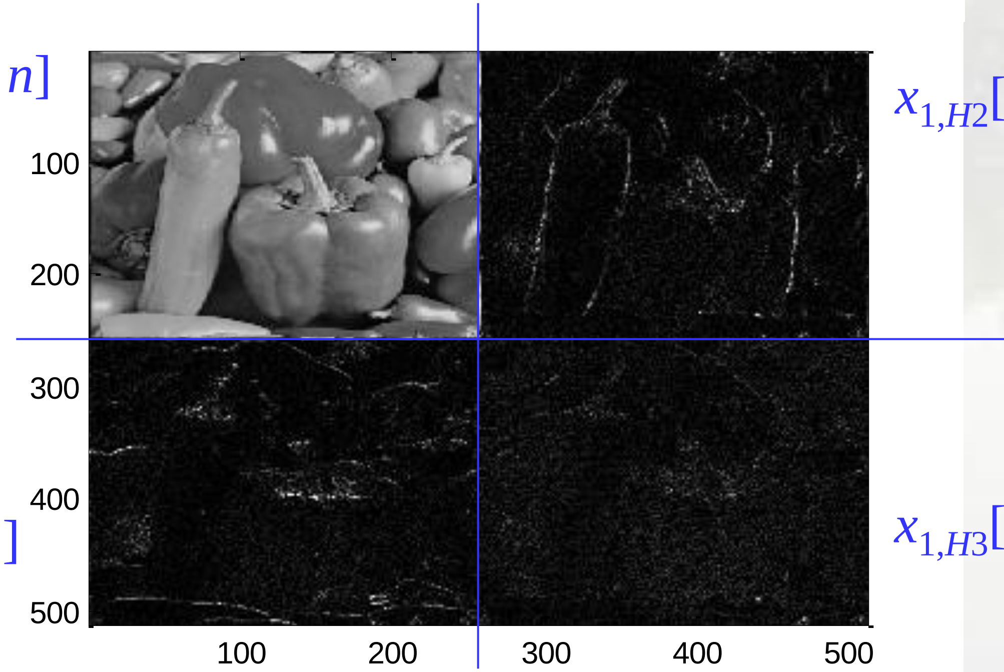

L-points
g[n]
↓2
along n
M × N
x[m, n]
L-points
h[n]
↓2
along n
v1,L[m, n]
v1,H[m, n]
g[m]
↓2
along m
x1,L[m, n]
m 低頻, n 低頻
h[m]
↓2
along m
x1,H1[m, n]
m 高頻, n 低頻
g[m]
↓2
along m
x1,H2[m, n]
m 低頻, n 高頻
h[m]
↓2
along m
x1,H3[m, n]
m 高頻, n 高頻
n
m
x[m,n]

原影像

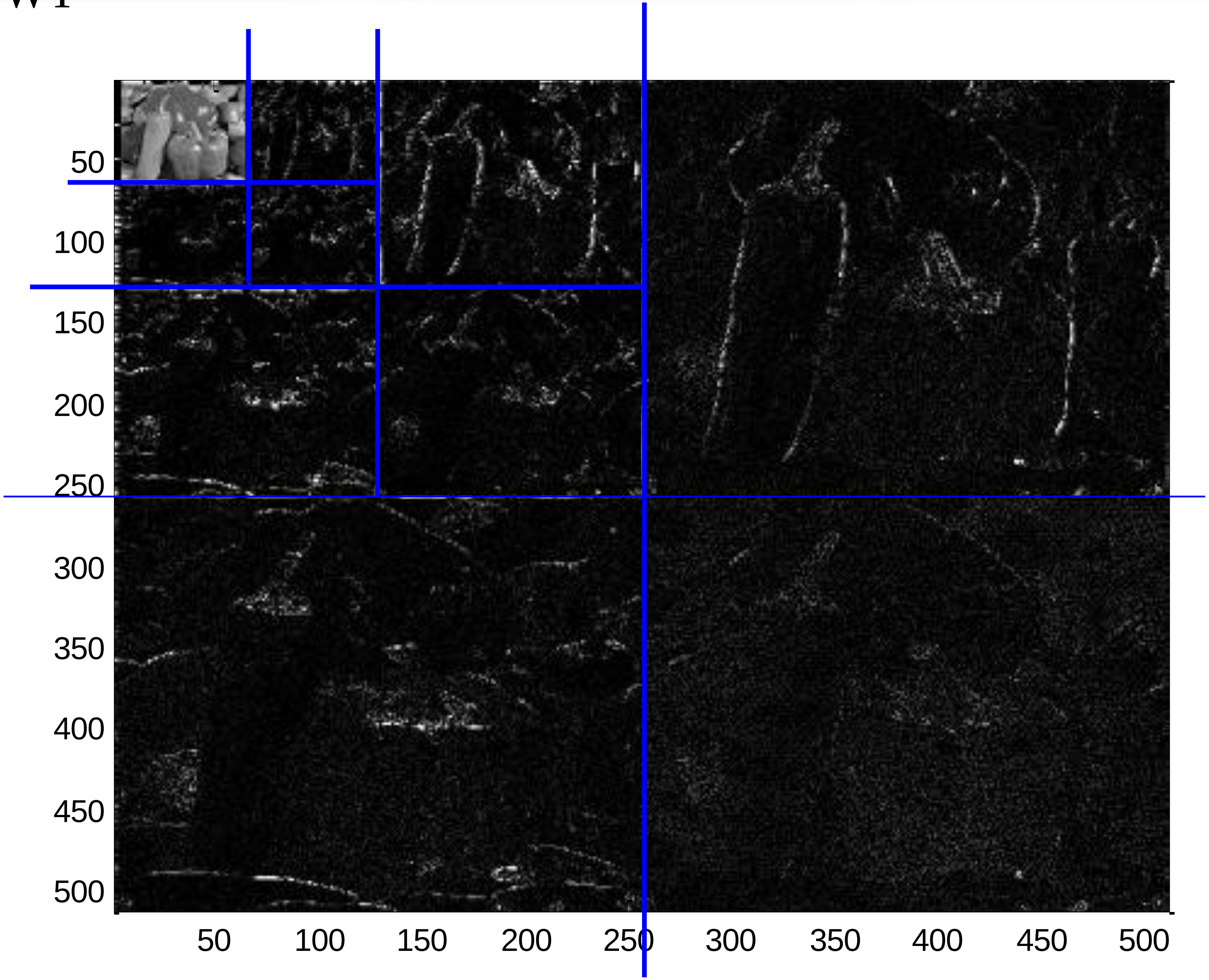


2-D DWT

的結果



3次2-D DWT 的結果


50
100
150
200
250
300
350
400
450
500
50
100
150
200
250
300
350
400
450
500

三、時頻分析近年來的發展

(1) Problem about Computation Time

(2) New Time-Frequency Analysis Tool

S transform

Asymmetric short-time Fourier transform

Gabor-Wigner transform

Directional Wavelet transform

Hilbert-Huang transform

(3) New Applications

Adaptive sampling theory

Adaptive filter design

Biology

3-1 Problem about Computation Time

(1) 對於許多信號的時頻分佈而言

$X(t,f)$ 和 $X(t+\Delta_t, f)$ 之間有高度的相關性

↓

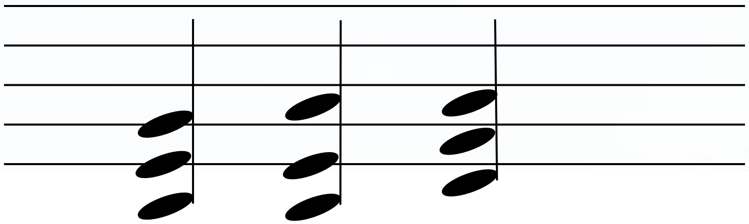
"Adaptive interval" and "interpolation"

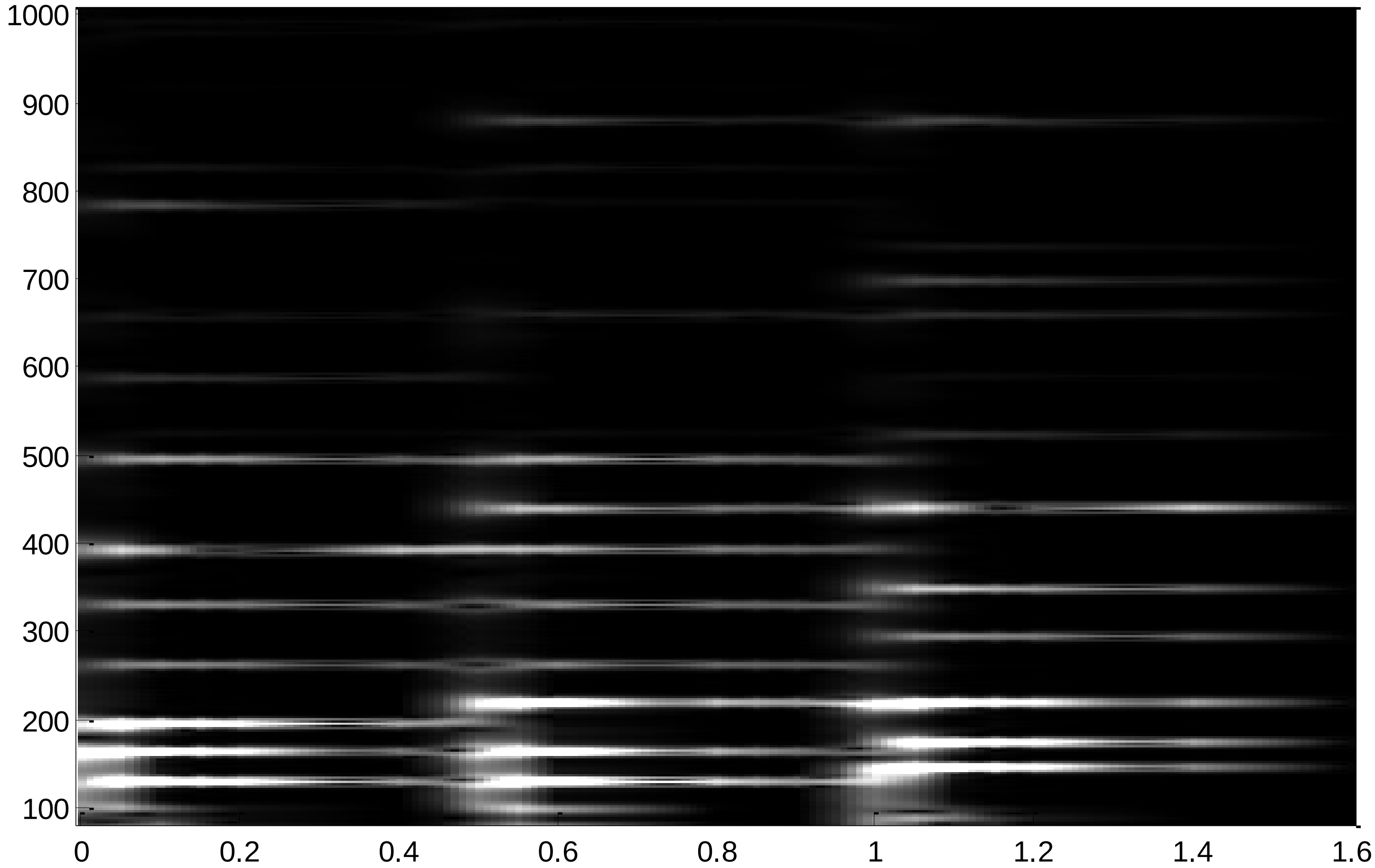
(2) 可預測瞬時頻率的位置

大部分信號瞬時頻率都偏低頻

且只要是由樂器或生物聲帶產生的信號，都會有「倍頻」的現象

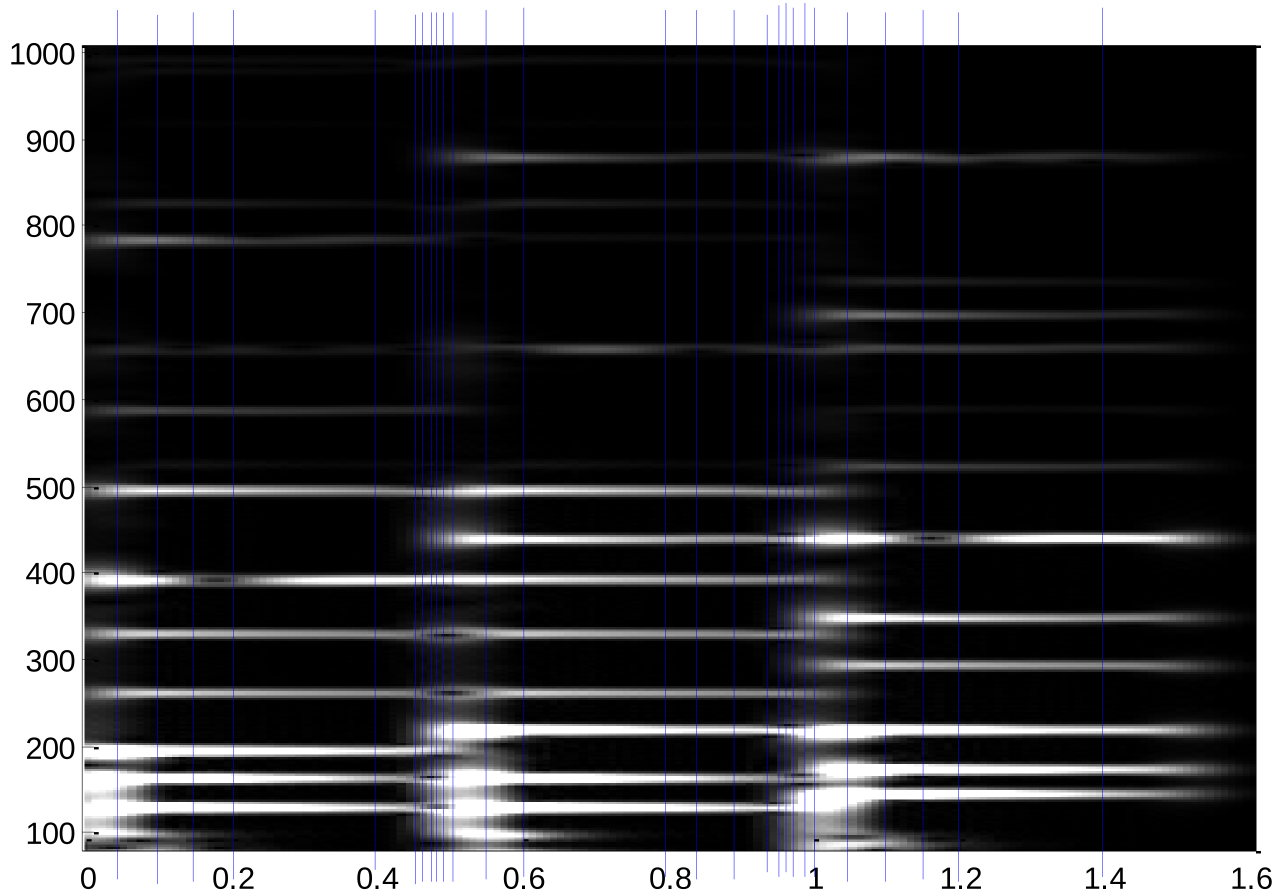
Short-time Fourier transform
of a music signal





Δ_τ = 1/44100 (總共有 44100 × 1.6077 sec + 1 = 70902 點)

with adaptive output sampling intervals



時頻分析的大家族

- (1) Short-time Fourier transform (STFT) (rec-STFT, Gabor, …)
  - square → spectrogram
  - improve → Asymmetric STFT
  - improve → S transform
  - combine → Gabor-Wigner Transform
- (2) Wigner distribution function (WDF)
  - combine → Gabor-Wigner Transform
  - improve → windowed WDF
  - improve → Cohen's Class Distribution (Choi-Williams, Cone-Shape, Page, Levin, Kirkwood, Born-Jordan, …)
  - improve → Pseudo L-Wigner Distribution
- (3) Wavelet transform
  - → Haar and Daubechies
  - → Coiflet, Morlet
  - → Directional Wavelet Transform
- (4) Time-Variant Basis Expansion
  - → Matching Pursuit
  - → Prolate Spheroidal Wave Function
- (5) Hilbert-Huang Transform (唯一跳脫 Fourier transform 的架構)

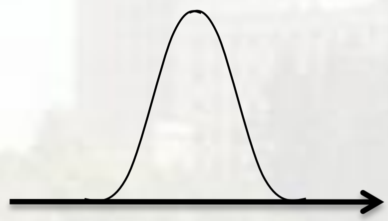
3-2 Asymmetric Short-Time Fourier Transform

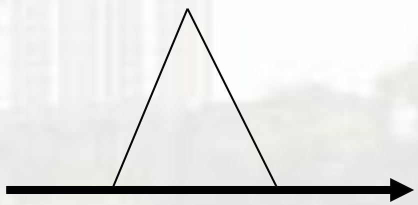
(2005 ?)

Short-Time Fourier Transform

$$X(t,f)=\int_{-\infty}^{\infty} w(t-\tau)x(\tau)e^{-j2\pi f\tau}d\tau$$

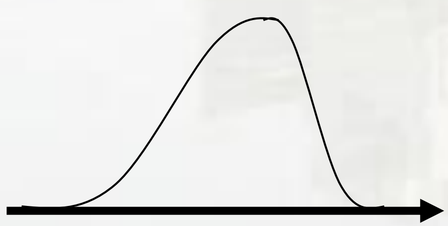
通常 $w(t)$ 是左右對稱的





但是在某些應用 (例如地震波的偵測)

使用非對稱的 window 會有較好的效果



3-3 S Transform (1996)

$$S_x(t,f) = |f| \int_{-\infty}^{\infty} x(\tau) \exp\left[-\pi (t-\tau)^2 f^2\right] \exp(-j2\pi f \tau)\, d\tau$$

比較：原本的 short-time Fourier transform 當 $w(t) = \exp(-\pi t^2)$ 時

$$X(t,f) = \int_{-\infty}^{\infty} x(\tau) \exp\left[-\pi t^2\right] \exp(-j2\pi f \tau)\, d\tau$$

$f\uparrow$, window width $\downarrow$

$f\downarrow$, window width $\uparrow$

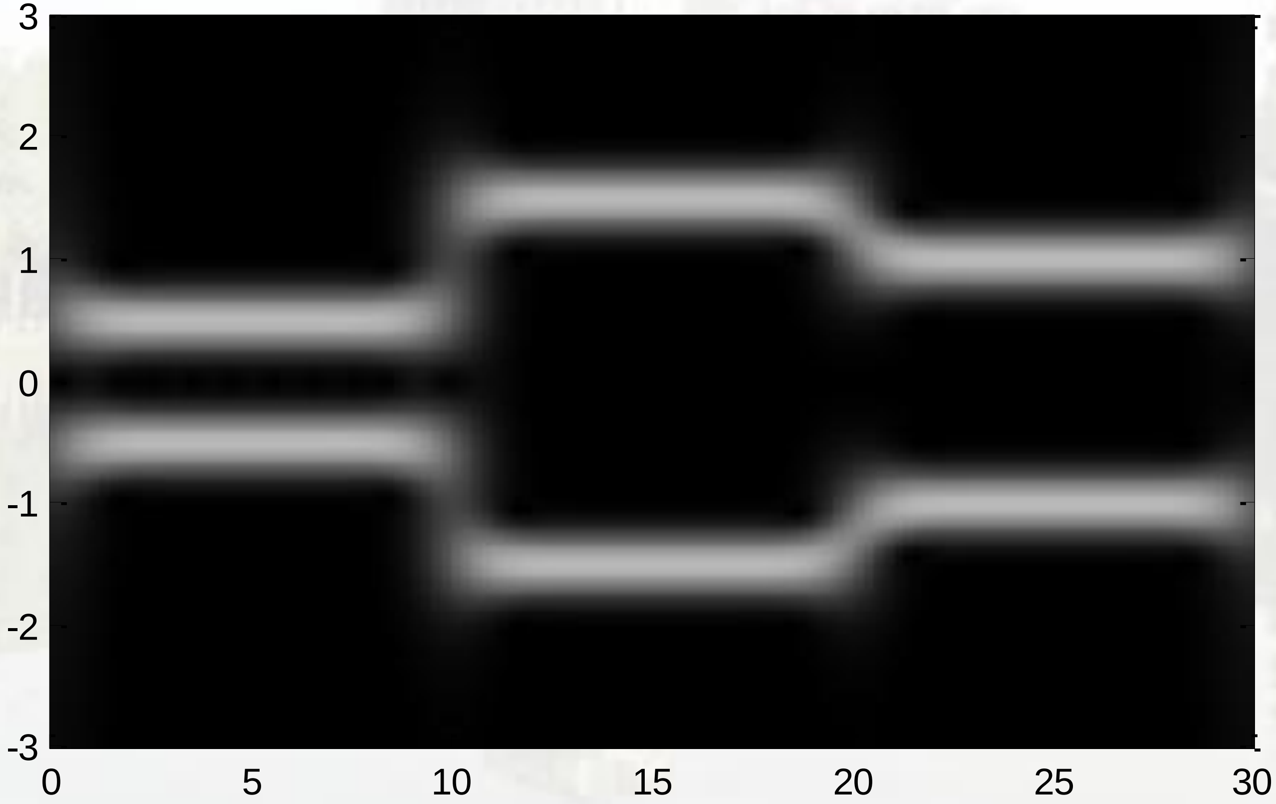
[Ref] R. G. Stockwell, L. Mansinha, and R. P. Lowe, "Localization of the complex spectrum: the S transform," *IEEE Trans. Signal Processing*, vol. 44, no. 4, pp. 998–1001, Apr. 1996.

$x(t) = \cos(\pi t)$ when $t < 10$,

$x(t) = \cos(3\pi t)$ when $10 \le t < 20$,

$x(t) = \cos(2\pi t)$ when $t \ge 20$

Using the short-time Fourier transform

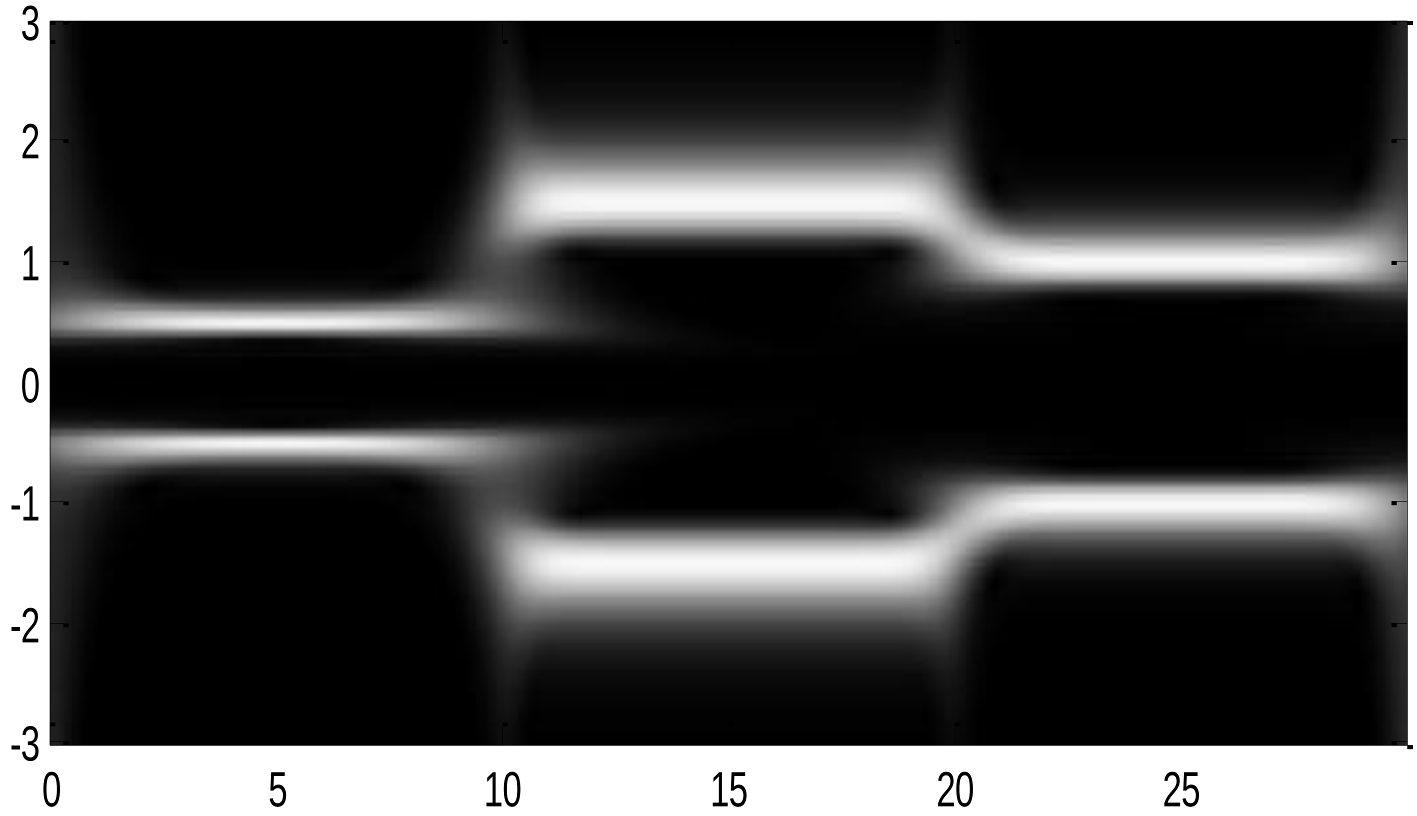


$x(t) = \cos(\pi t)$ when $t < 10$,

$x(t) = \cos(3\pi t)$ when $10 \le t < 20$,

$x(t) = \cos(2\pi t)$ when $t \ge 20$

Using the S transform



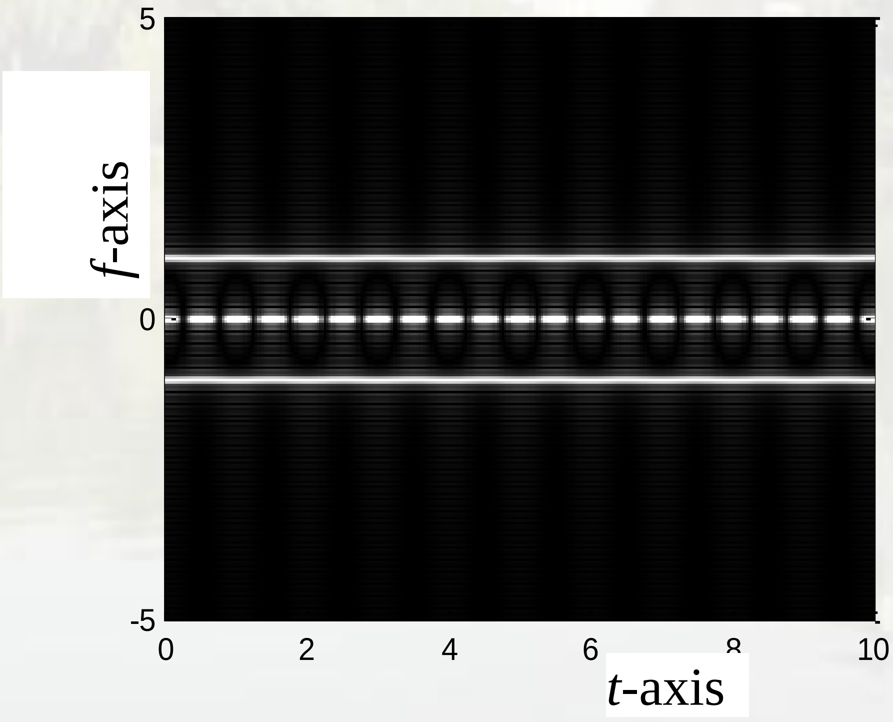
3-4 Gabor-Wigner Transform (2007)

如何同時達成 (1) high clarity

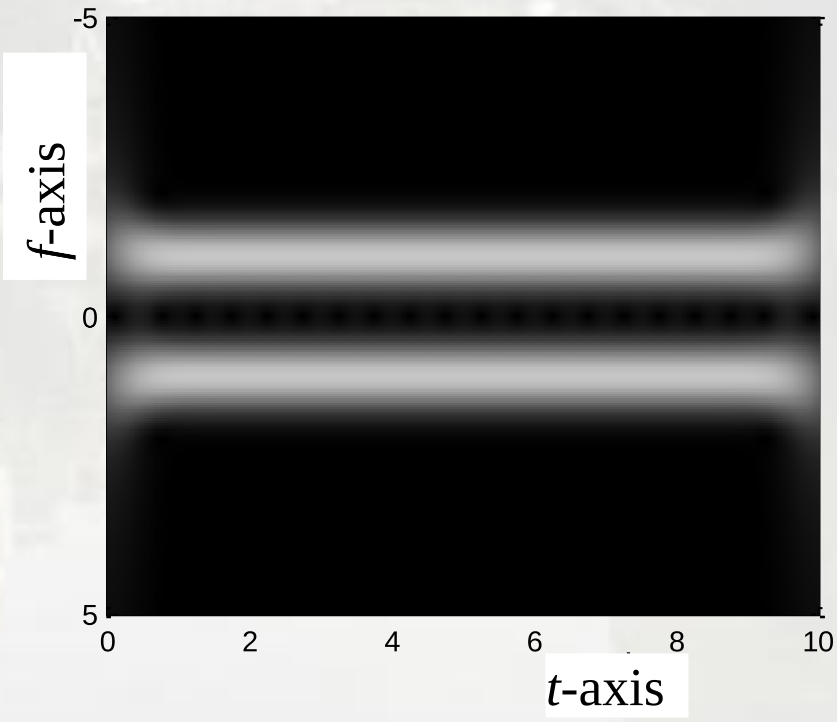
(2) no cross-term 的目標？

$\cos(2\pi t)$

by WDF



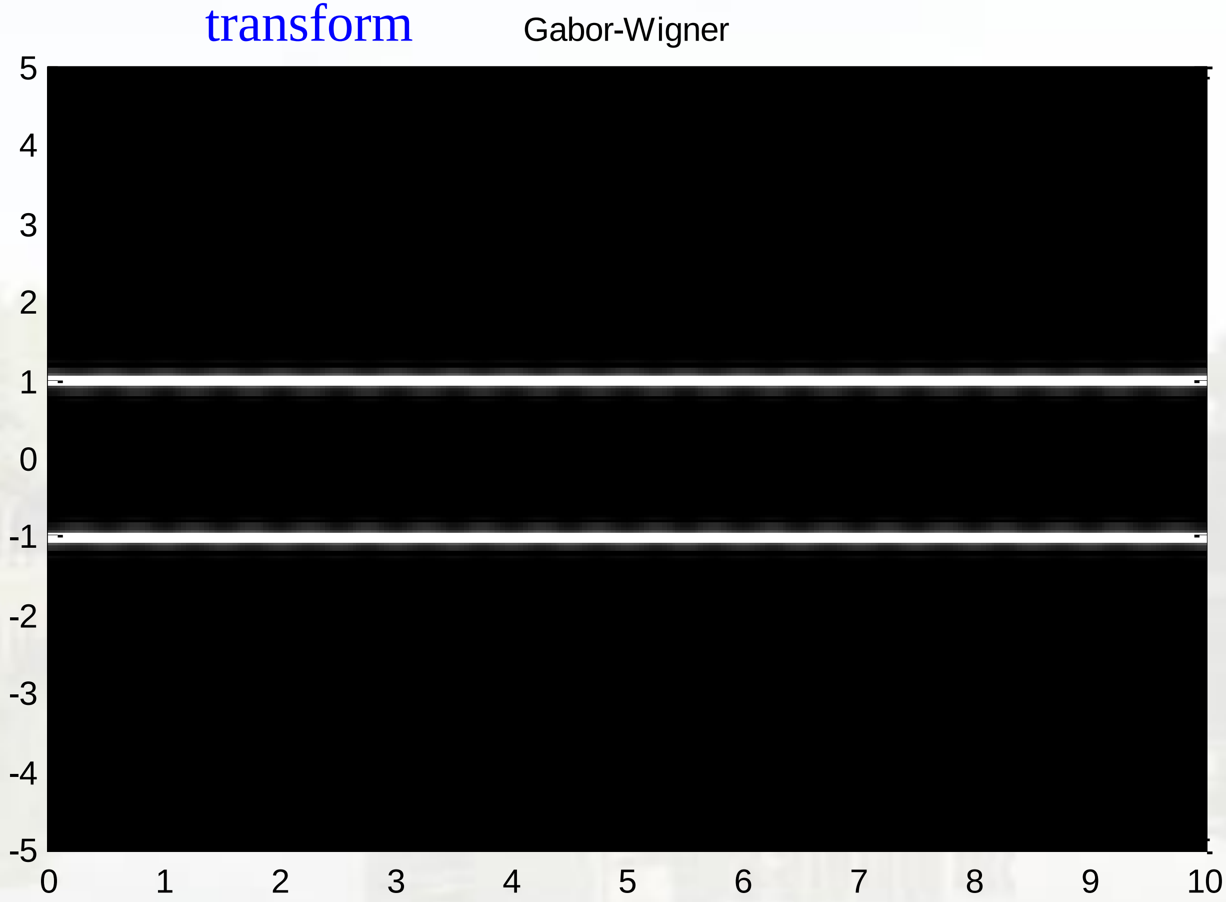
by short-time Fourier transform



$$D_x(t,\omega) = G_x(t,\omega)W_x^2(t,\omega)$$

Short-time Fourier transform

Wigner



[Ref] S. C. Pei and J. J. Ding, "Relations between Gabor transforms and fractional Fourier transforms and their applications for signal processing," *IEEE Trans. Signal Processing*, vol. 55, no. 10, pp. 4839-4850, Oct. 2007.

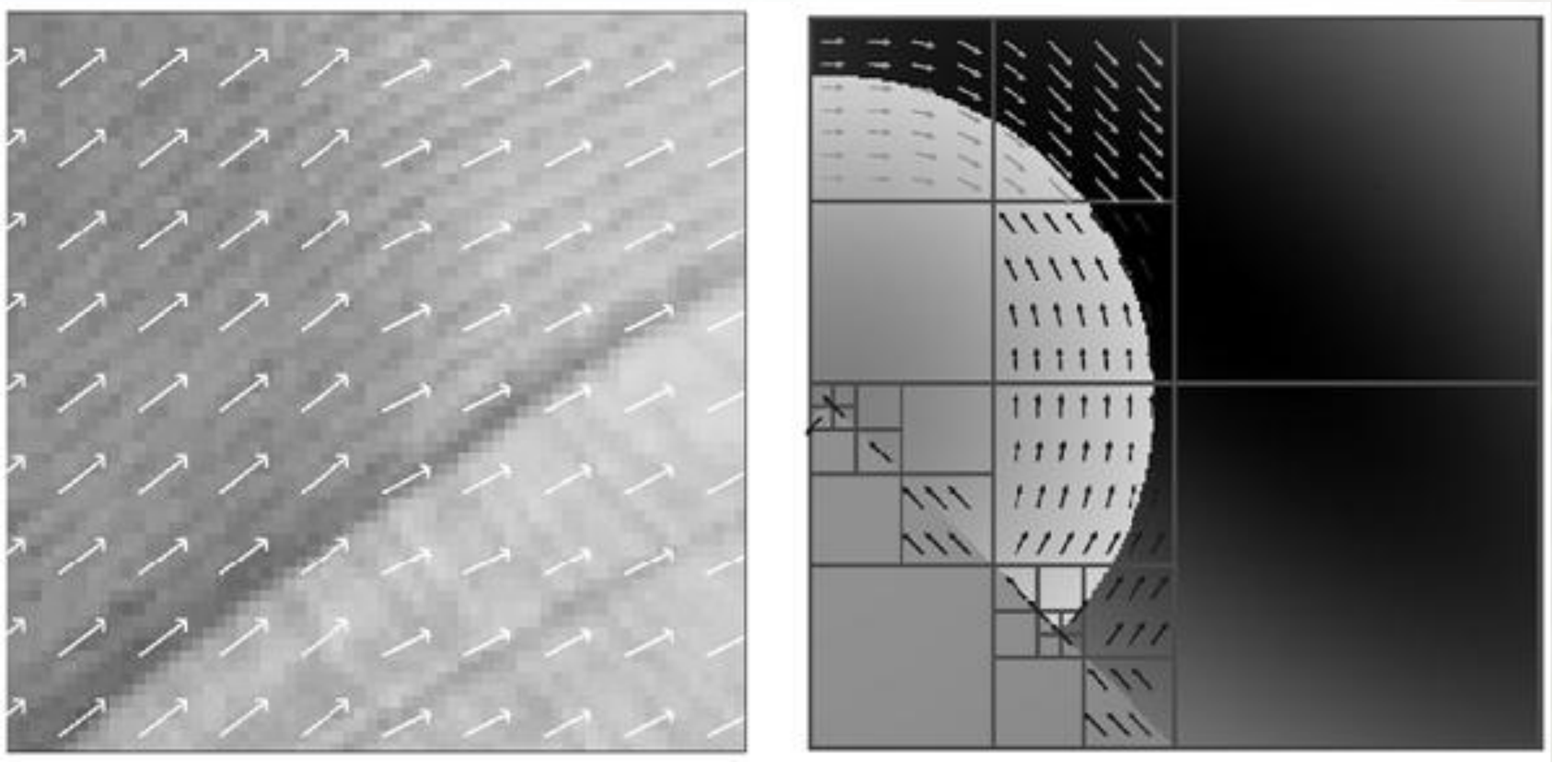
3-5 Directional Wavelet Transform (1999)

Wavelet transform 未必要沿著 *x*, *y* 軸來做

- curvelet
- contourlet
- bandlet
- shearlet
- Fresnelet
- wedgelet
- brushlet

- **Bandlet**

根據物體的紋理或邊界，來調整 wavelet transforms 的方向



Stephane Mallet and Gabriel Peyre, "A review of Bandlet methods for geometrical image representation," *Numerical Algorithms*, Apr. 2002.

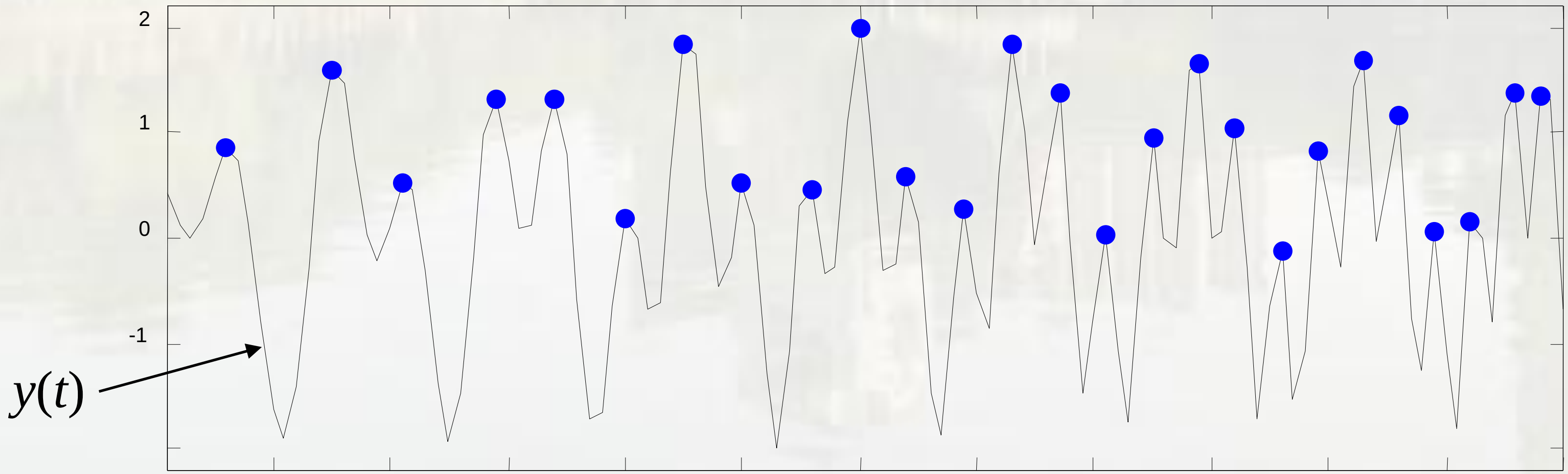
3-6 Hilbert-Huang Transform (國產) (1998)

為中研院黃鍔院士於 1998 年提出

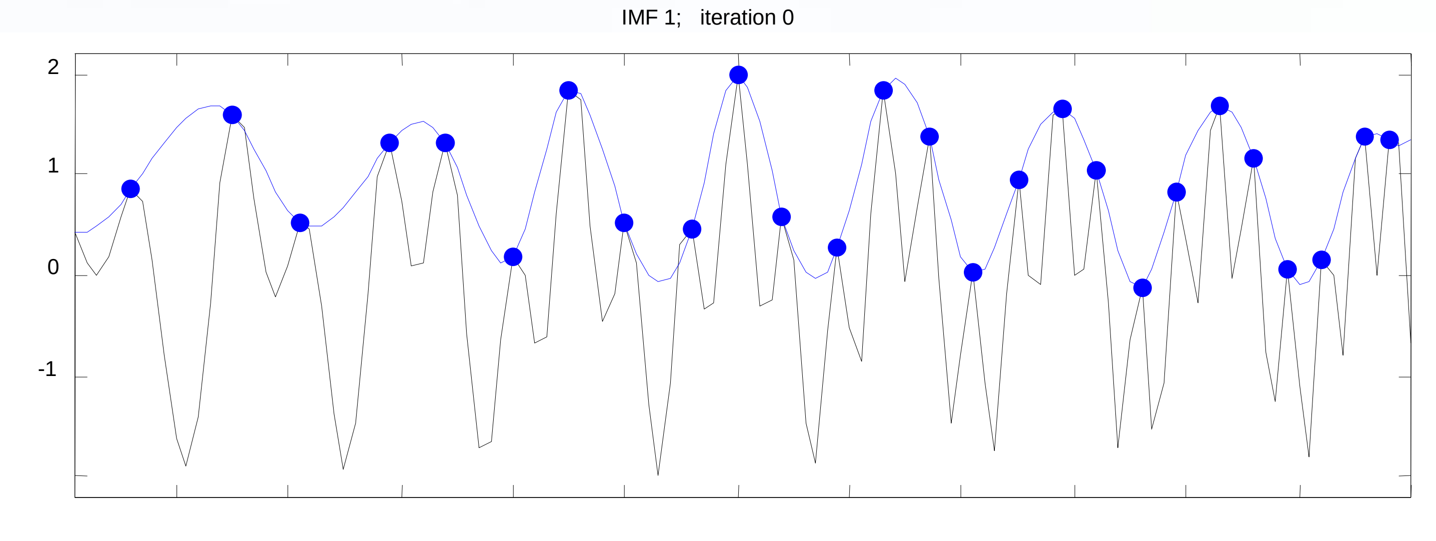
時頻分析，何必要用到那麼複雜的數學？

(Step 1) Initial: $y(t) = x(t)$, ($x(t)$ is the input) $n = 1$, $k = 1$

(Step 2) Find the local peaks



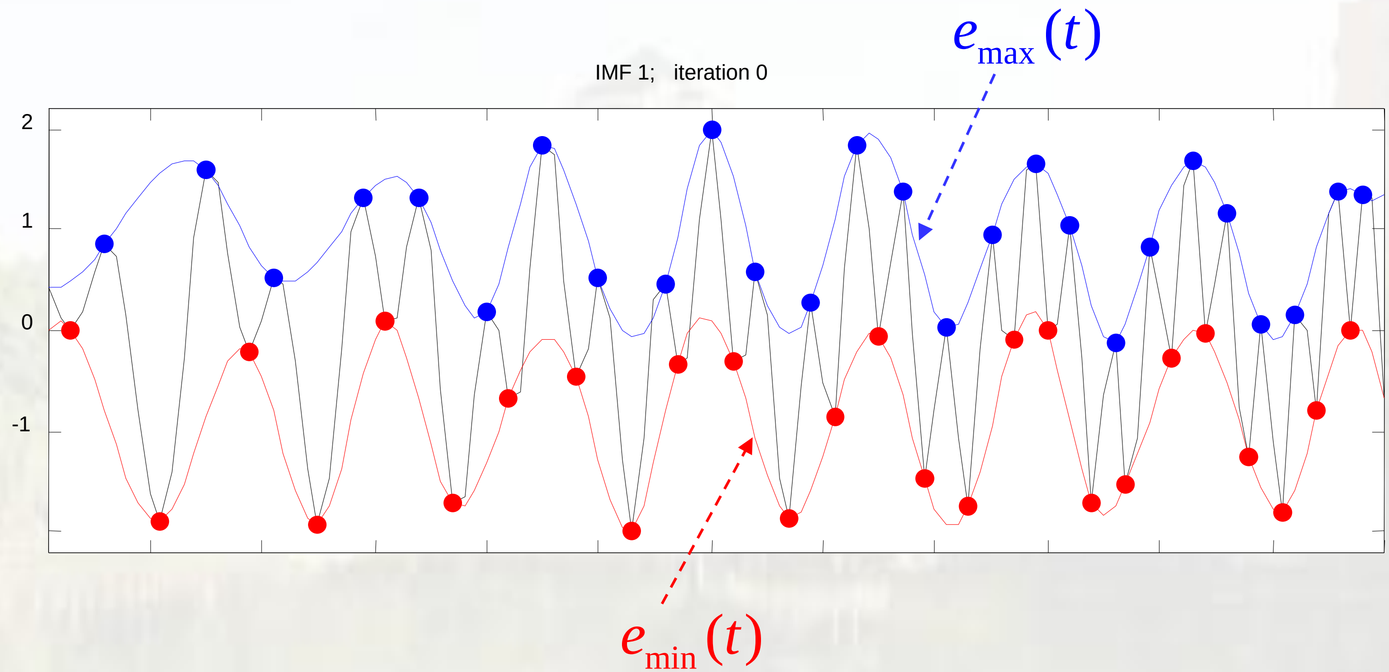
(Step 3) Connect local peaks



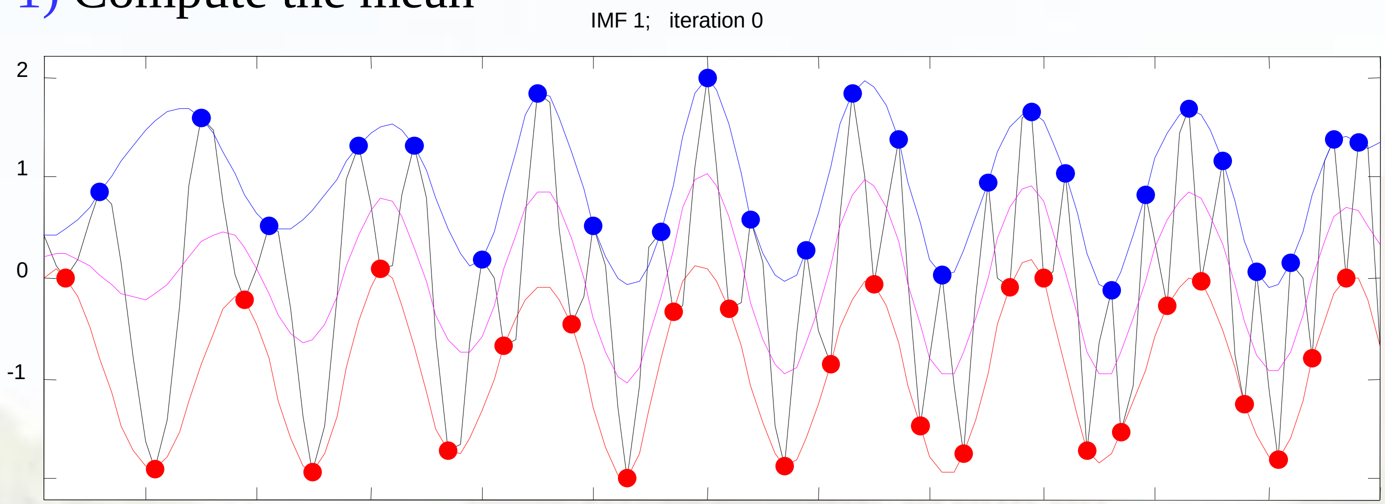
通常使用 B-spline，尤其是 cubic B-spline 來連接

(Step 4) Find the local dips

(Step 5) Connect the local dips

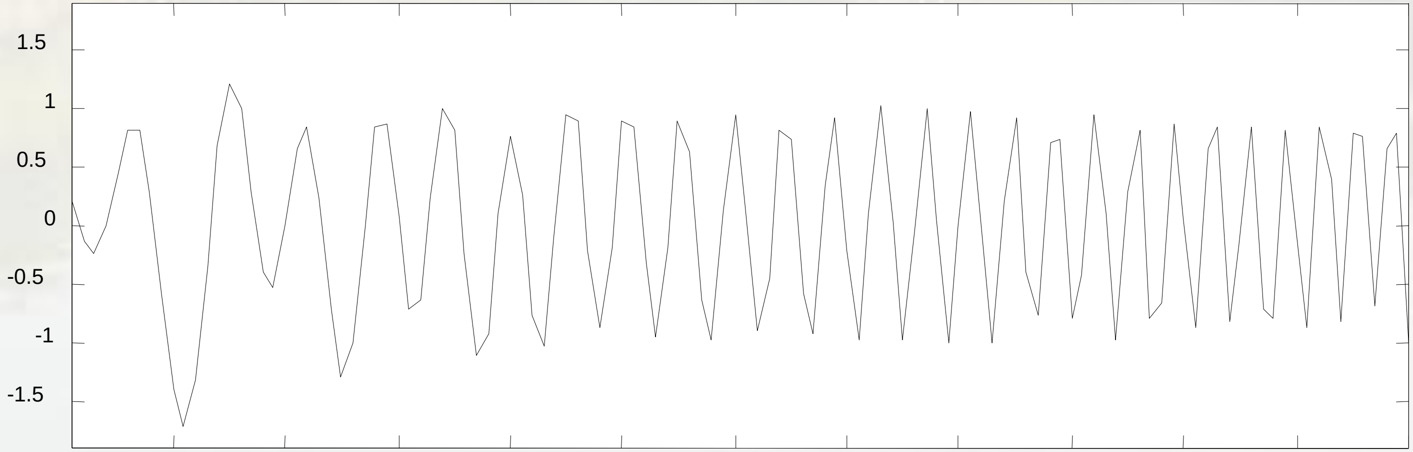


(Step 6-1) Compute the mean



$$z(t) = \frac{e_{\min}(t) + e_{\max}(t)}{2} \quad \text{(pink line)}$$

(Step 6-2) Compute the residue $h_k(t) = y(t) - z(t)$



Step 7 Repeat Steps 1-6 to determine the intrinsic mode function (IMF)

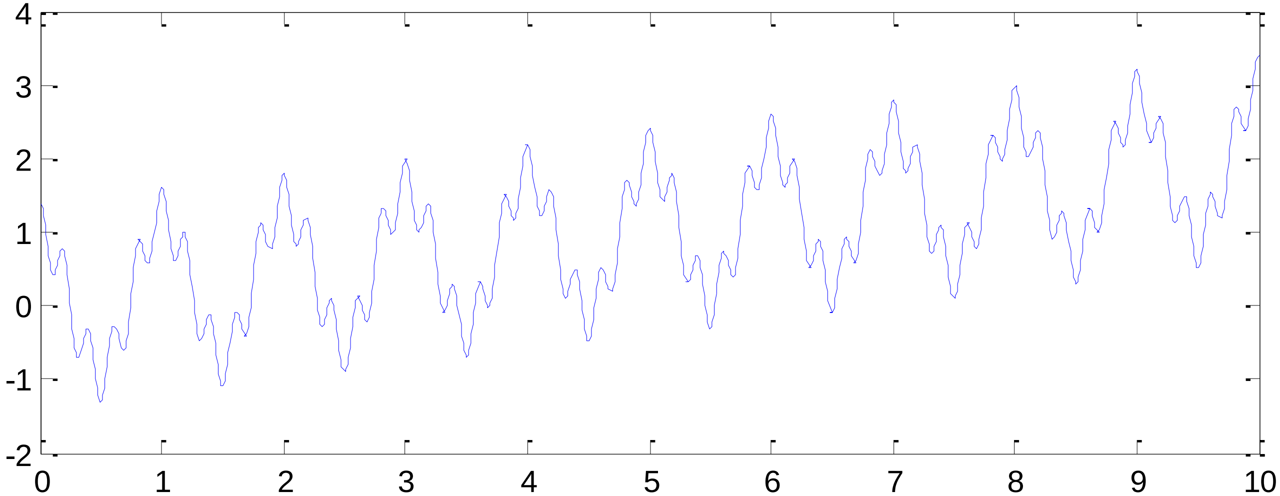
Step 8 Repeat Steps 1-7 to further determine $x(t)$

Step 9 Determine the instantaneous frequency for each IMF

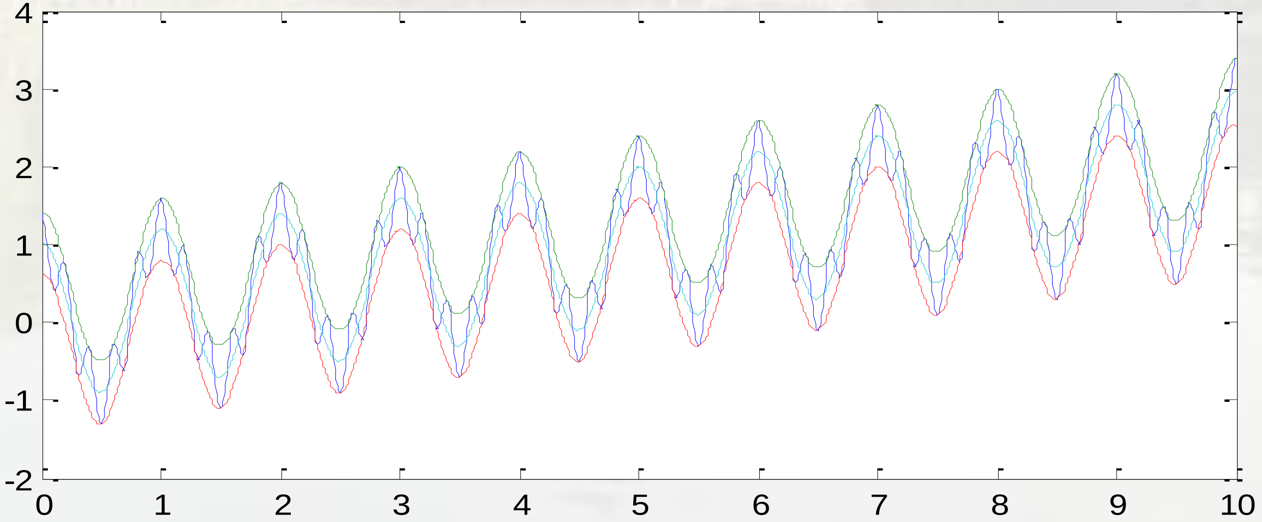
(just calculating the number of zero-crossing during [t-1/2, t+1/2] and dividing the result by 2)

Example

$$x(t) = 0.2t + \cos(2\pi t) + 0.4\cos(10\pi t)$$



After Step 6

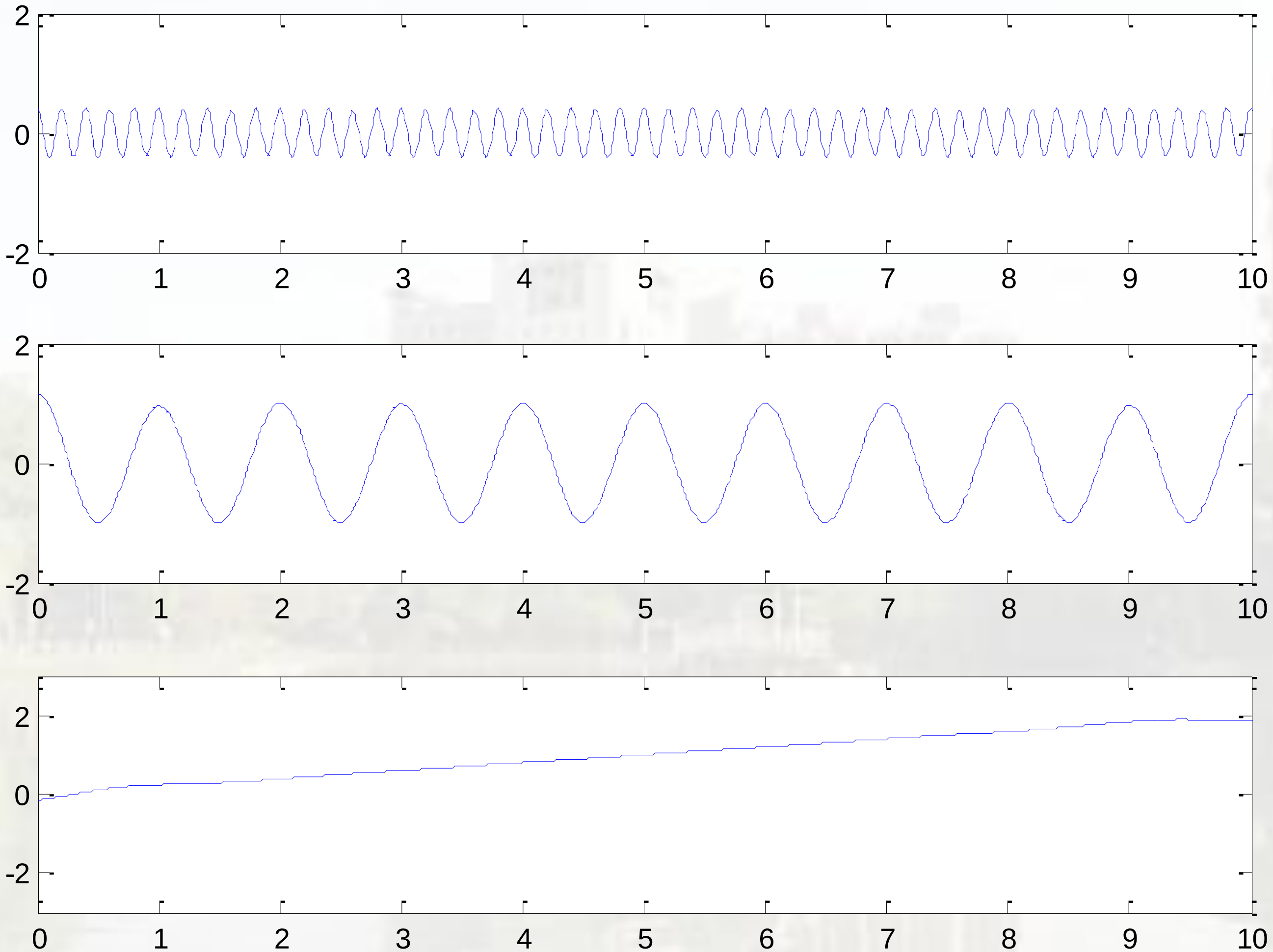


IMF1

IMF2

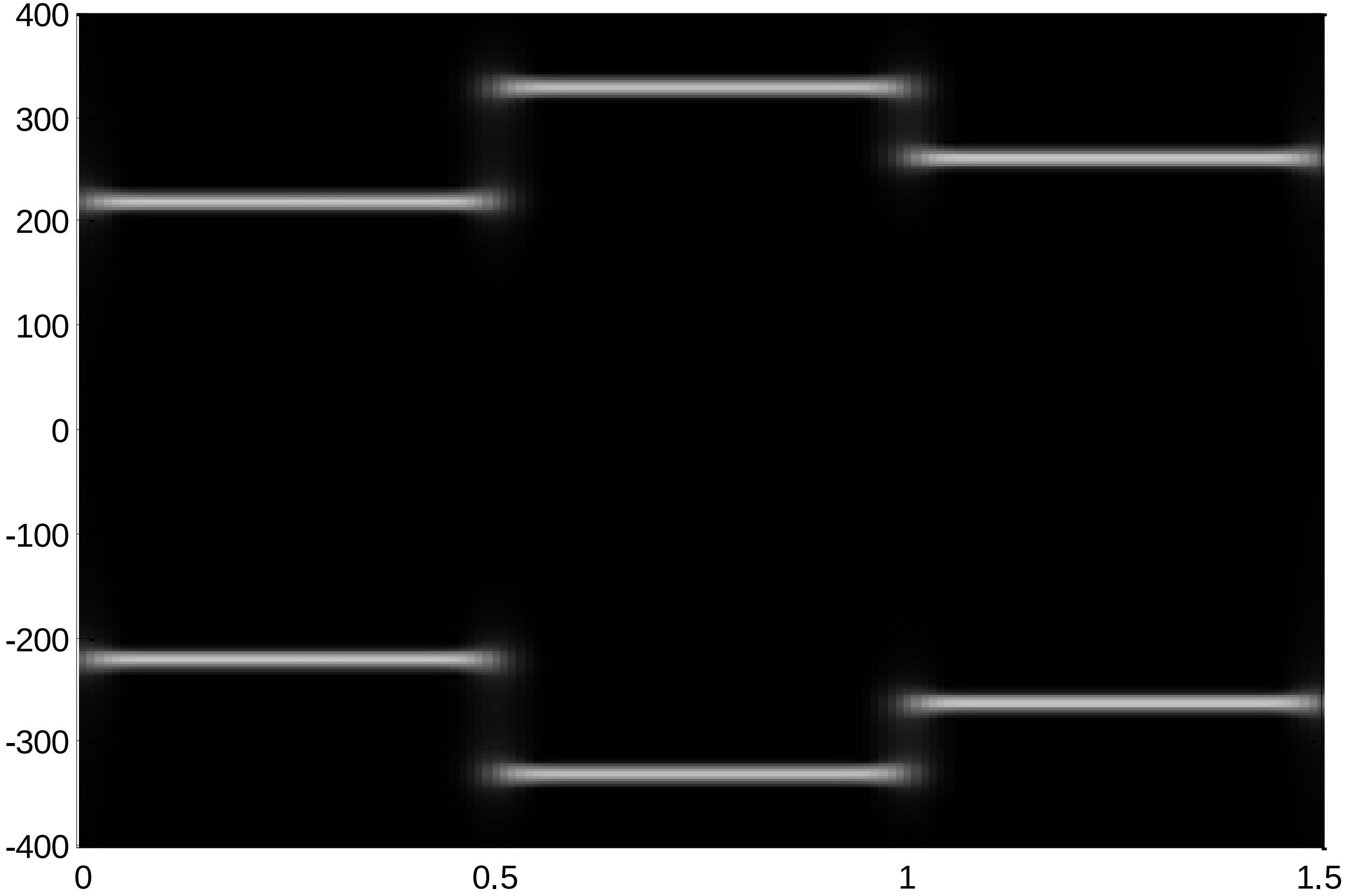
$x_0(t)$

趨勢



3-7 Application: Adaptive Sampling

Nyquist rate: $\Delta_t < 1/2B$ B: bandwidth

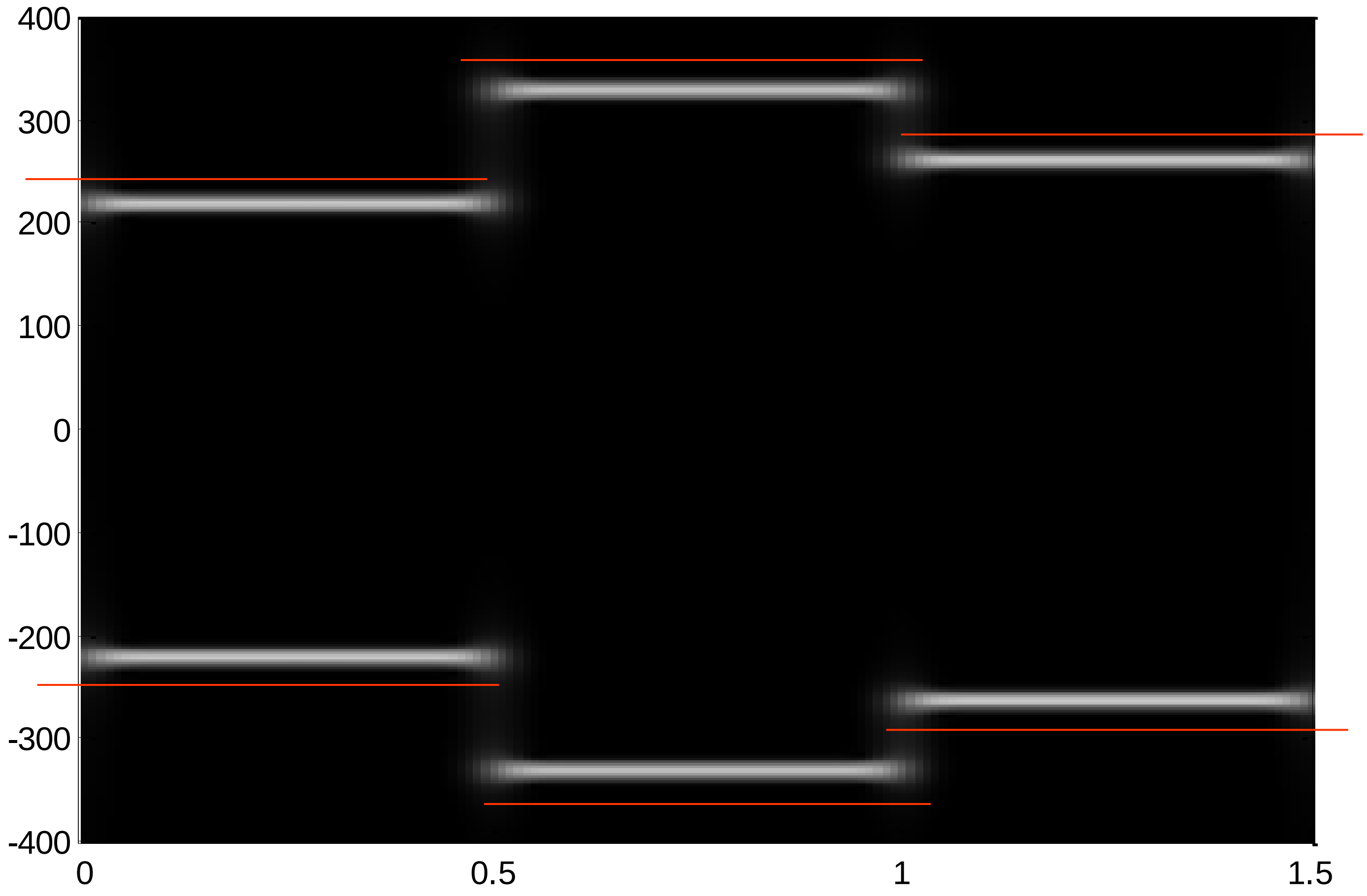


重要定理：

Number of sampling points == Area of time frequency distribution

3-8 Application: Adaptive Filter Design

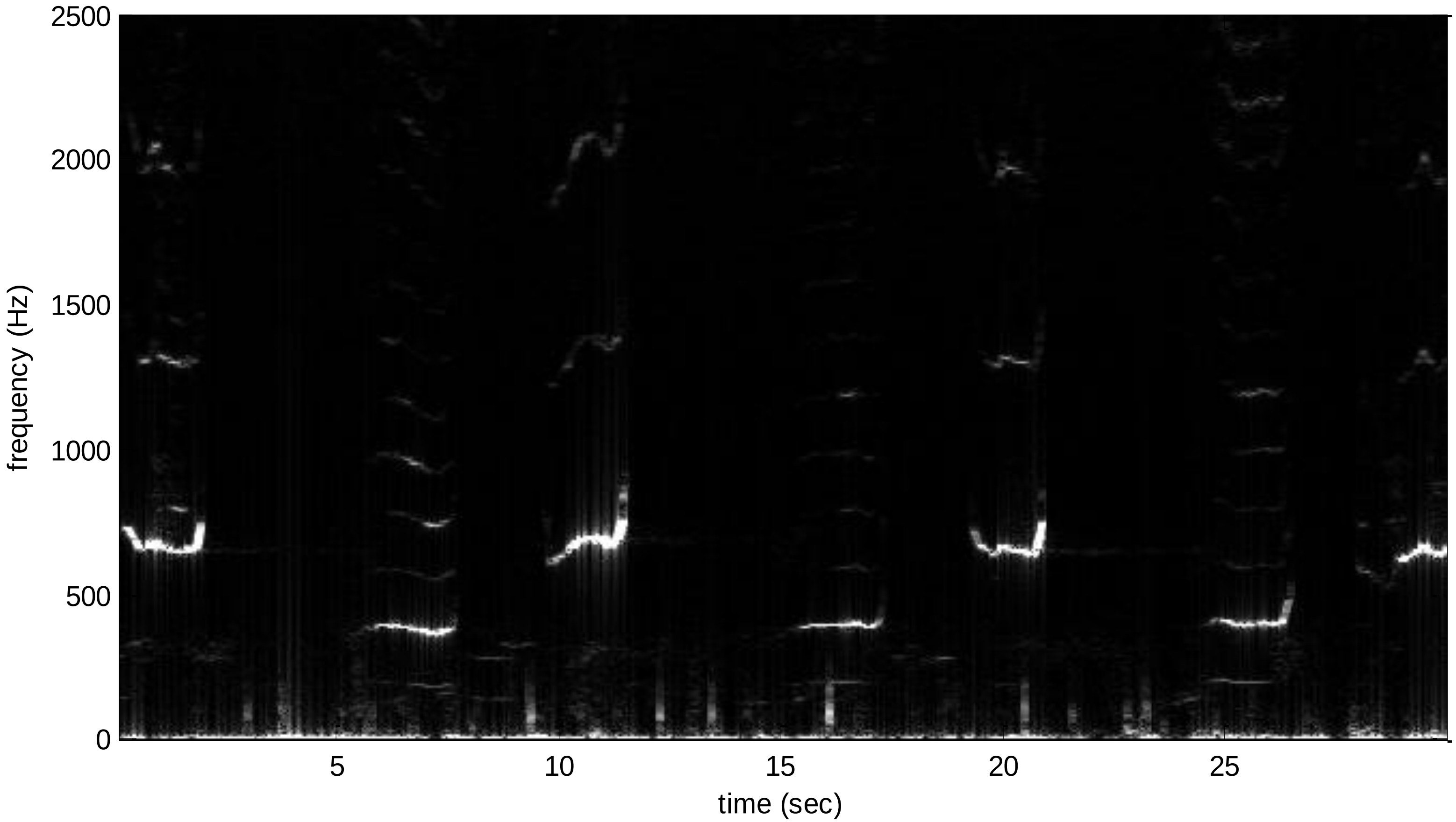
Adaptive cutoff



3-9 Applications in Biology

Whale voice

Data source: http://oalib.hlsresearch.com/Whales/index.html



四、結論

(1) 由於計算速度的大幅提升，使得使用「時頻分析」來取代「傅立葉轉換」來做信號分析變得更加可行

所有傅立葉轉換的應用，都將會是時頻分析的應用

(2) 時頻分析新理論和新應用的發展，有待大家共同努力

投影片下載網址： http://djj.ee.ntu.edu.tw/TF.ppt