Conceptual Signal Modeling: The Idea and its Applications in Groundwater Analysis

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Contents

- Signal modeling for groundwater
  - Lumped systems of groundwater
  - Aquifer system analysis
  - Size specific lumped system
  - Pumping recovery strength
- Numerical experiments
  - Parameters discussion
  - Numerical experiments
  - Fluid balance analysis
  - Groundwater in mountain area
- Conclusion

SIGNAL MODELING FOR GROUNDWATER

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Signal modeling is to represent single or multiple signals via some model parameters.

Signal modeling is used for signal prediction, reconstruction and understanding.

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Why Conceptual Signal Modeling?

$$\frac{\partial u}{\partial t} + \hat{L}_x u = f(x,t) \qquad u = u(x,t)$$

- Partial differential equation
- I.C. => unknown
- B.C. => unknown
- Source terms => unknown
- Material parameters => unknown

Conceptual Signal Modeling

$$\frac{\partial u}{\partial t} + P(u,t) = f(t) \qquad u = u(t)$$

- Ordinary differential equation
- No spatial operator
- I.C. => measurement data
- Material parameters => fitting

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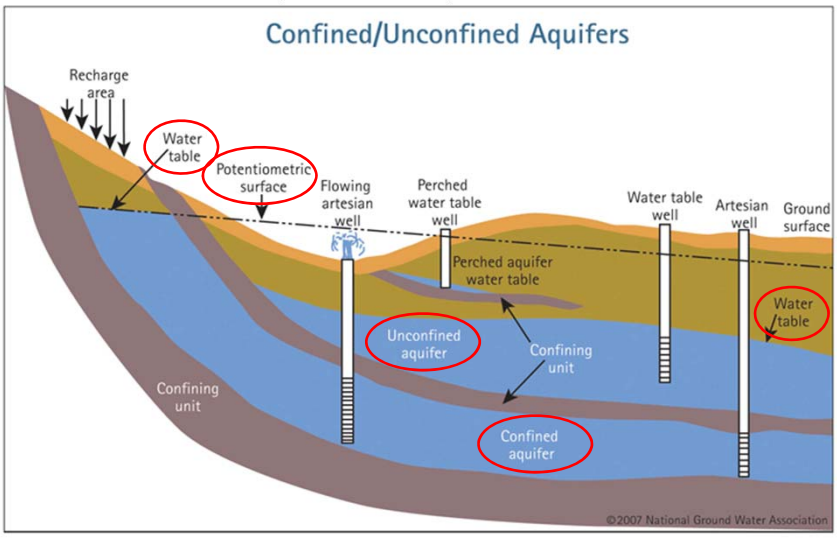
Aquifer Systems



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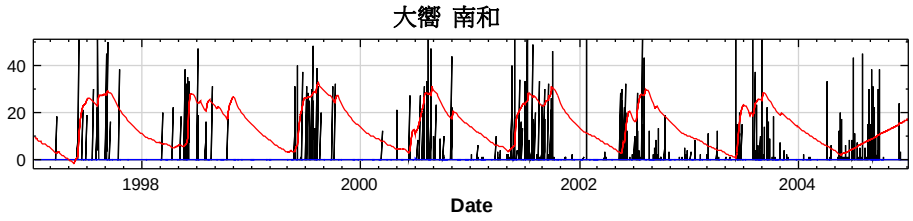
Lumped Systems of Groundwater



- The level will gradually decay in dry season.
- The variation of groundwater level is highly related to rainfall.

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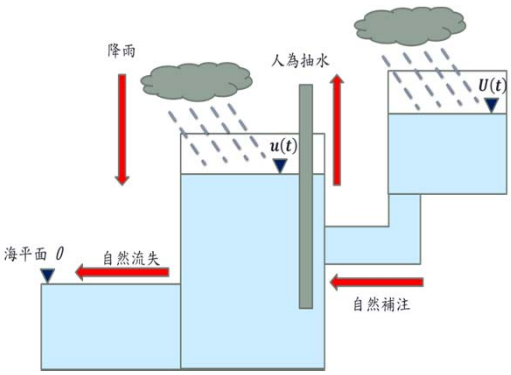
Lumped Systems of Groundwater

- Input:
  - Natural recharge
- Output:
  - Natural discharge
- Groundwater level:
  - Not varies spatially
  - Only a function of time



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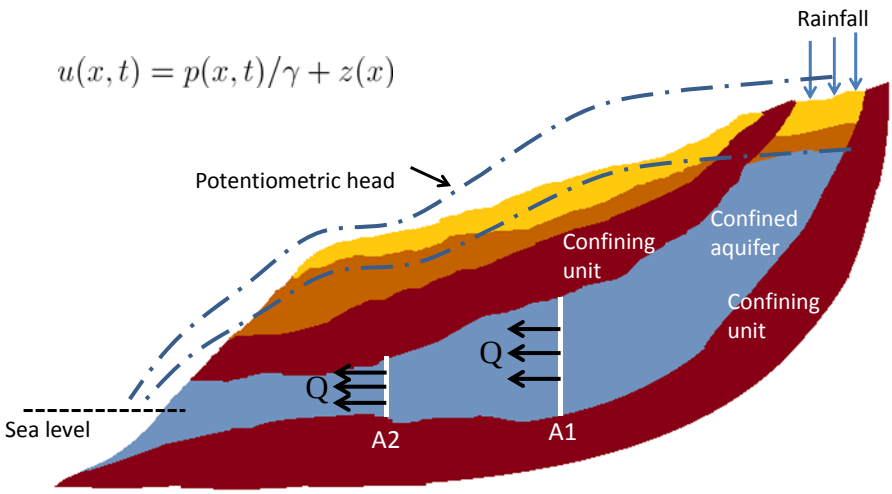
Confined Aquifer System

$$u(x,t) = p(x,t)/\gamma + z(x)$$



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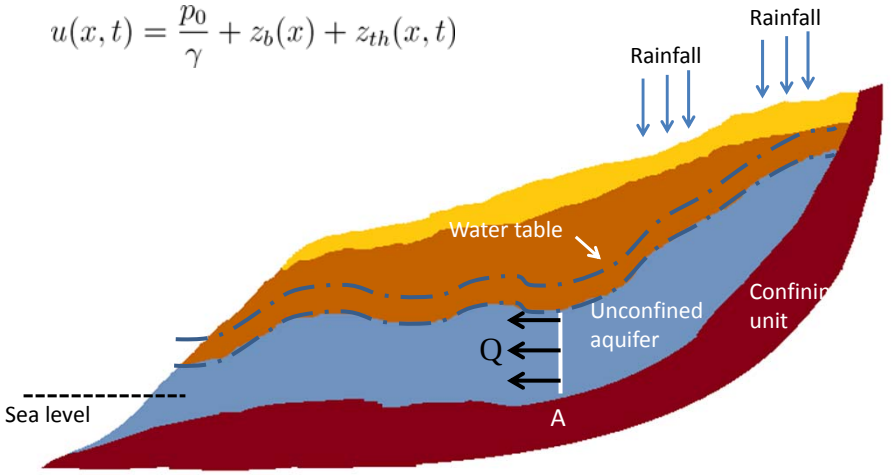
Unconfined Aquifer System

$$u(x,t) = \frac{p_0}{\gamma} + z_b(x) + z_{th}(x,t)$$



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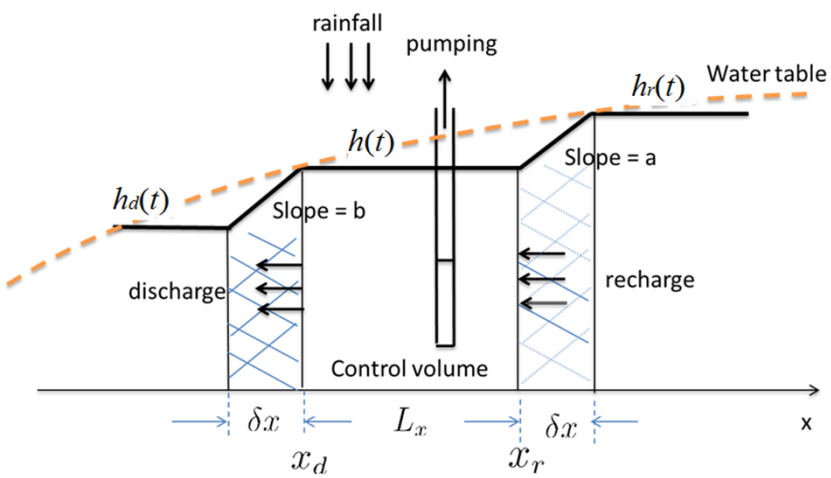
Assumptions

- No vertical flow (Dupuit Assumption)
- Size specific lumped system
- Quasi-static

Lumped System(unconfined aquifer)



Schematic illustration of size specific lumped system.

Size Specific Lumped System

- Source free equation for unconfined aquifer:

$$Q_r = -KL_y u(t) \cdot a \qquad a = \frac{\partial h}{\partial x}, \quad \text{at} \quad x = x_r$$

$$Q_d = -KL_y u(t) \cdot b \qquad b = \frac{\partial h}{\partial x}, \quad \text{at} \quad x = x_d$$

$$L_y L_x S_y \frac{\partial u}{\partial t} = -KL_y(a+b)u(t)$$

  – Where L_x is length of lumped system along the axis of flow, L_y is the length of flow area in y-direction, and S_y is the specific yield.

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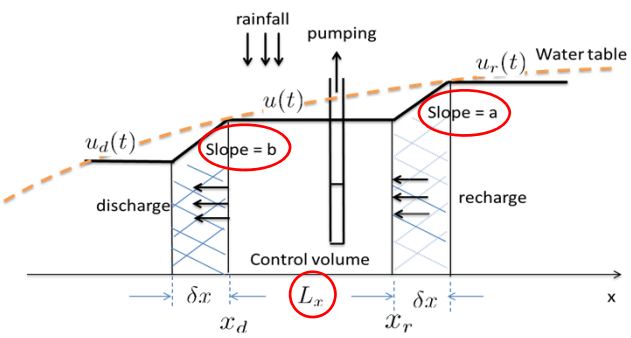
Size Specific Lumped System

- We can rearrange the equation as $\frac{\partial u}{\partial t} = -\lambda u$
- Dissipating coefficient: $\lambda = \frac{K(a+b)}{S_y L_x} \sim \frac{K}{S_y}$



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Size Specific Lumped System

- Consider pumping and rainfall effect

$$\frac{\partial u^*}{\partial t} = -\lambda u^*(t) + \gamma R(t) - \sigma P(t)$$

  – Where $R(t)$ and $P(t)$ represent rainfall infiltration and artificial pumping respectively.
  – The associated parameters γ and σ relates to rainfall infiltration and pumping equivalency.
  – Pumping effect can be approximated via PRS (pumping recovery strength)

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Lumped Systems of Groundwater

- Three major parameters:
  – Dissipation coefficient
    - The time constant of how long water be kept in the system.
  – Rainfall Infiltration coefficient
    - Convert a fraction of observed precipitation effectively injected into groundwater reservoir.
  – Pumping conversion coefficient
    - Convert a fraction of observed pumping effectively out of groundwater reservoir.
- The parameters can be identified with the aid of historical data.

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Size Specific Lumped System

- Consider the bottom level , where $u^*(t) = u(t) - u_b$,and the solution is

$$u(t) = u_b(1 - e^{-\lambda t}) + e^{-\lambda t}[u_0 + \int_0^t e^{\lambda\tau}(\gamma R(\tau) - \sigma P(\tau))d\tau]$$

- Since the equation is linear, we can evaluate the solution with respect to constant and periodical sink or source respectively.

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Case I: Source free condition

- Without infiltration and pumping, the solution suggests water level decay as an exponential function, and approaches u_b.
- The trend suggests groundwater acts more like a reservoir as it retains water easily when located closer to the seashore.

$$u(t) = u_b + (u_0 - u_b)e^{-\lambda t}$$

Half-time period: $t_{\frac{1}{2}} = \frac{\ln 2}{\lambda}$

	Lambda (1/year) [normalized]	Half-time period /phase delay(days)
大嚮/南和	3.9 [0.62]	64.9/59
萬隆/新來義	1.1 [0.175]	229/81.2
赤山/泰武	0.7 [0.111]	361/84.8
瑪家/三地門	0.6 [0.095]	421/85.7

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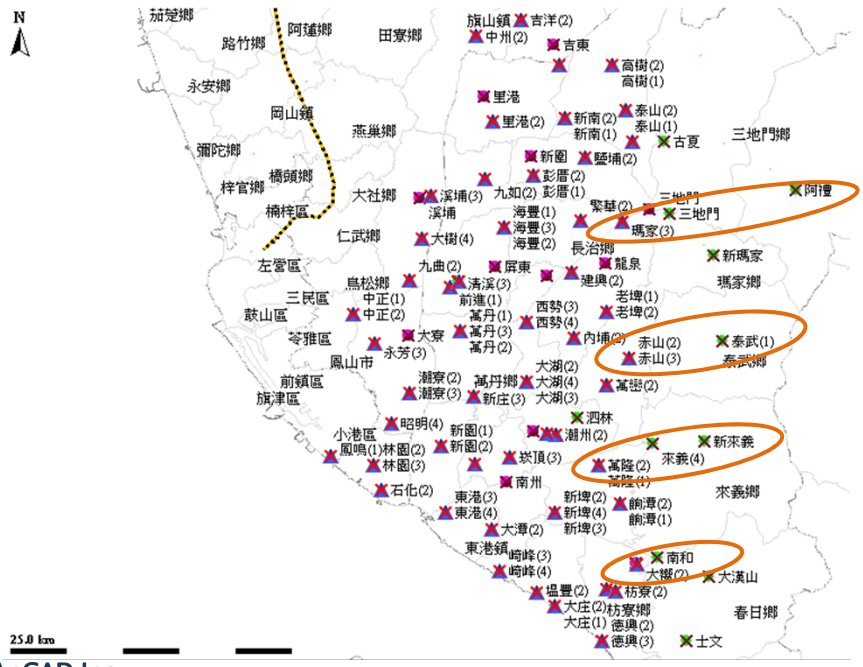

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Case II: Constant source or sink

- Artificial pumping might create constant sink by doing continuous pumping.
- For area of large dissipating coefficient, the variation due to constant pumping is relatively small as can be observed by taking time variable to infinity.

$$\frac{du^*}{dt} + \lambda u^* = C$$

$$u = u_b + (u_0 - u_b)e^{-\lambda t} + \frac{C}{\lambda}(1 - e^{-\lambda t})$$

- The result suggests restricted pumping be applied in area of lower dissipating coefficient such as in the vicinity of seashore.

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Case III: Periodical source

- Periodical sources in ground water system are mainly attributed to seasonal rainfall (1 year, RRS) and artificial pumping (1 day, PRS).
- RRS: Rainfall Recovery Strength
- PRS : Pumping Recovery Strength

$$\frac{du^*}{dt} + \lambda u^* = C\cos(\omega_p t)$$

$$u = u_b + (u_0 - u_b)e^{-\lambda t} + \frac{C}{\lambda^2 + \omega_p{}^2}(\lambda\cos\omega_p t + \omega_p \sin\omega_p t - \lambda e^{-\lambda t})$$

Normalized dissipation coefficient

$$\lambda_n = \frac{\lambda}{\omega_p}$$

And normalized frequency

$$\omega_n = \frac{\omega_p}{\lambda}$$

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$$u = u_b + e^{-\lambda t}(u_0 - u_b - \frac{\lambda C}{\lambda^2 + w_p^2}) + \frac{C}{\sqrt{\lambda^2 + w_p^2}}\cos(w_p t - \theta)$$

where the phase difference

$$\theta = \tan^{-1}\omega_n$$

$$\lambda = \frac{\omega_p}{\tan(t_p\omega_p)}$$

t_p is the delayed time between peak of rainfall to peak of groundwater.

$$\gamma = \frac{A_g}{A_r}\sqrt{\lambda^2 + \omega_p^2}$$

where A_g and A_r are the FFT magnitudes at 1 time/year of groundwater and rainfall.

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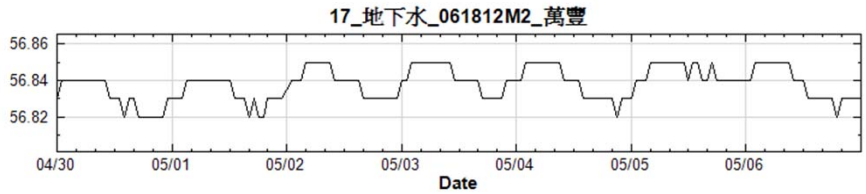
The Behavior of Artificial Pumping



- The main frequency component of artificial pumping is 1 time a day owing to the regular daily routine of human being.
- The PRS can be extracted from the fundamental frequency and its harmonics, which is a measure of short-period fluctuation riding on ground water level.
- Pumping strength (rate) is directly proportional to PRS.

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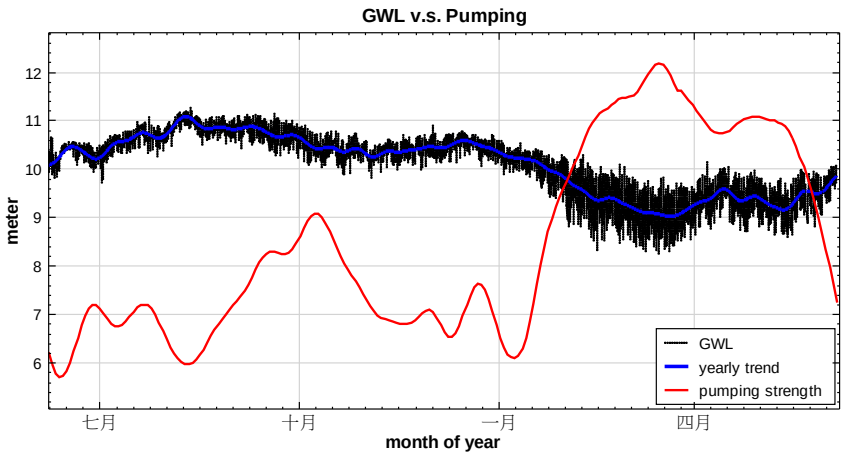
GWL vs. Pumping



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Pumping Recovery Strength

- Evaluate steps:
  1. Remove trending data of groundwater level.
  2. Using Notch filter to extract signal of pumping frequency (1 cpd) plus its harmonic components. PRS as a function of time results.
  3. Square PRS as a function of time to obtain its energy, and take integration to get total energy.
  4. Divide total energy by length of time to evaluate the average PRS energy.
  5. Take square root of the average PRS energy to get the average magnitude.

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NUMERICAL EXPERIMENTS

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Contents

- Parameters discussion
- Numerical experiments
- Fluid balance analysis
- Groundwater in mountain area

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Parameters Discussion

$$\frac{\partial u^*}{\partial t} = -\lambda u^*(t) + \gamma R(t) - \sigma P(t)$$

$$u(t) = u_b(1 - e^{-\lambda t}) + e^{-\lambda t}[u_0 + \int_0^t e^{\lambda \tau}(\gamma R(\tau) - \sigma P(\tau))d\tau]$$

What will the solution be if the parameters change?

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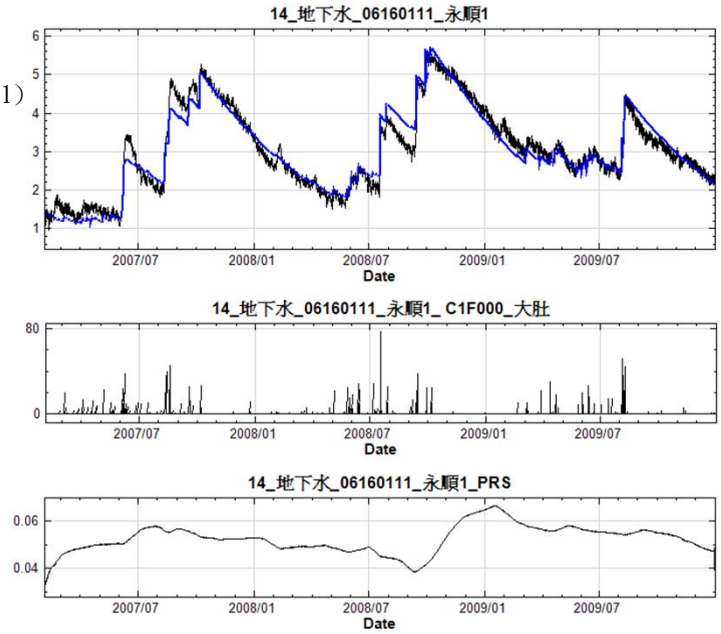
地下水測站：永順(1)
雨量站：大肚

$\lambda = 1.2633$
(1/year)
$\gamma = 86.7284$
$\sigma = 0.0748$
$u_b = -0.7682$ (m)



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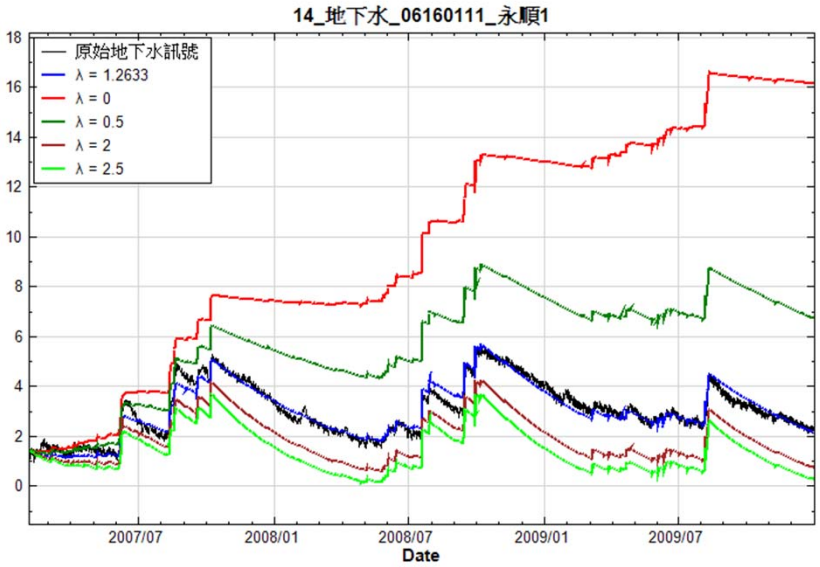

Fix γ, σ, u_b and change λ.

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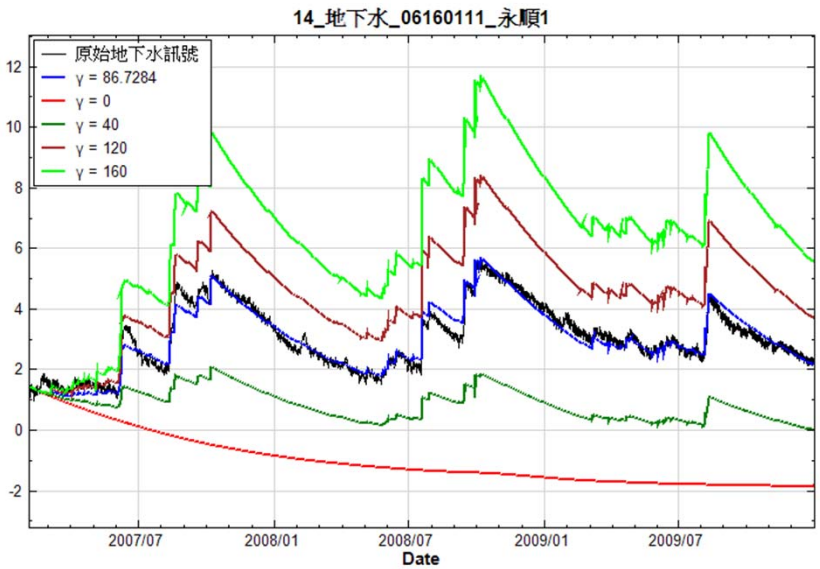

Fix λ, σ, u_b and change γ.

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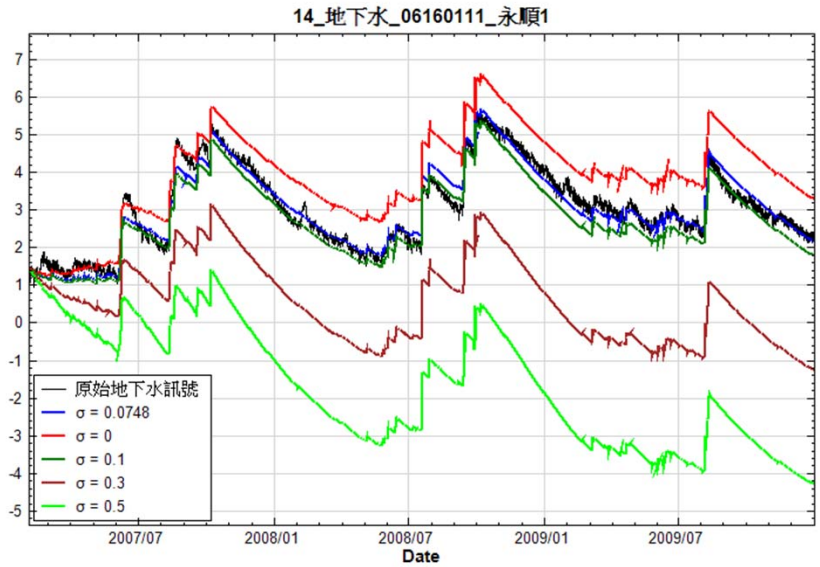

Fix λ, γ, u_b and change σ.

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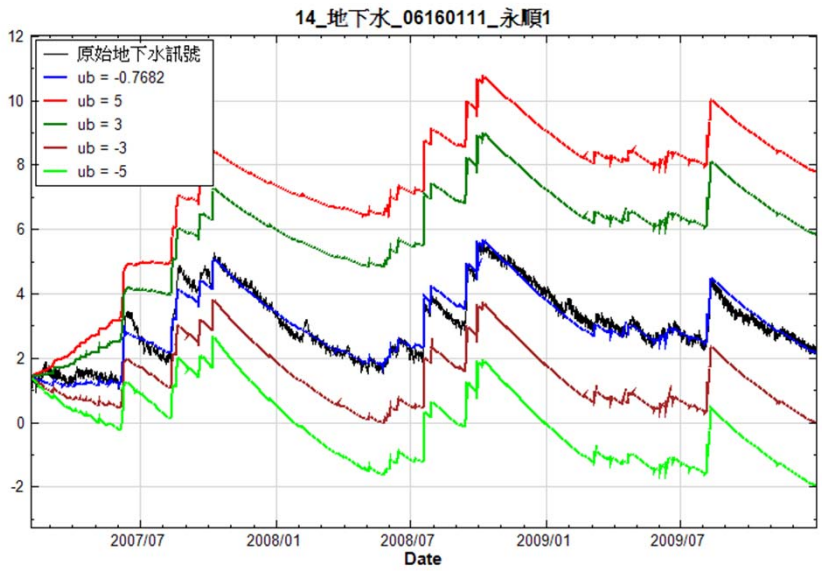

Fix λ, γ, σ and change u_b.

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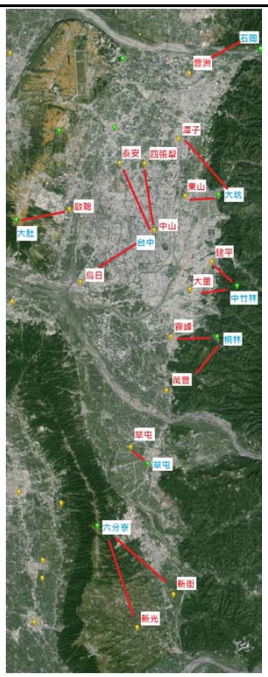
台中盆地地下水測站與雨量站分布

地下水位觀測井	雨量站
06080111_豐洲	C1F920_石岡
06090111_潭子	C1F970_大坑
06150111_烏日(1)	467490_台中
06180111_霧峰(1)	C1F9B0_桐林
061812M2_萬豐	C1F9B0_桐林
06190111_建平	C1F9A0_中竹林
06200111_大里(1)	C1F9A0_中竹林
08030111_草屯	01H630_草屯
08060111_新光(1)	01H590_六分寮
08060311_新街(1)	01H590_六分寮
19010111_中山(1)	467490_台中
19060111_啟聰(1)	C1F000_大肚
19060211_泰安	467490_台中
19080111_四張犁(1)	467490_台中
190812M2_東山	C1F970_大坑



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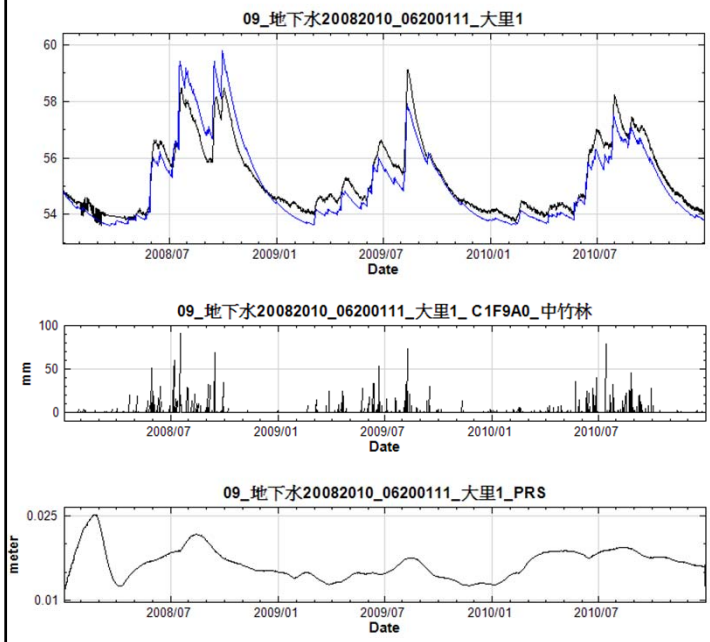

地下水測站：大里(1)
雨量站：中竹林

λ =7.2180 (1/year)
γ =131.4693
σ = 0.4637
u_b = 53.6 (m)

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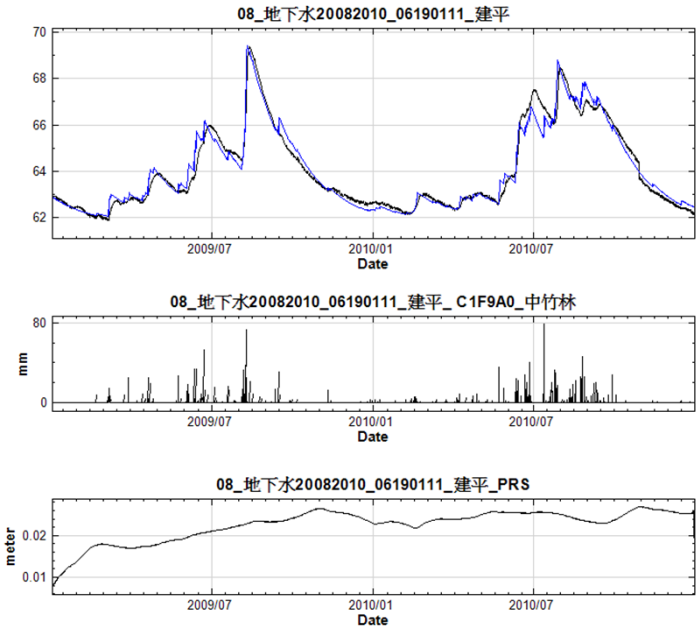

地下水測站：建平
雨量站：中竹林

λ =8.0773 (1/year)
γ =227.9148
σ = 0.1000
u_b = 61.7471 (m)

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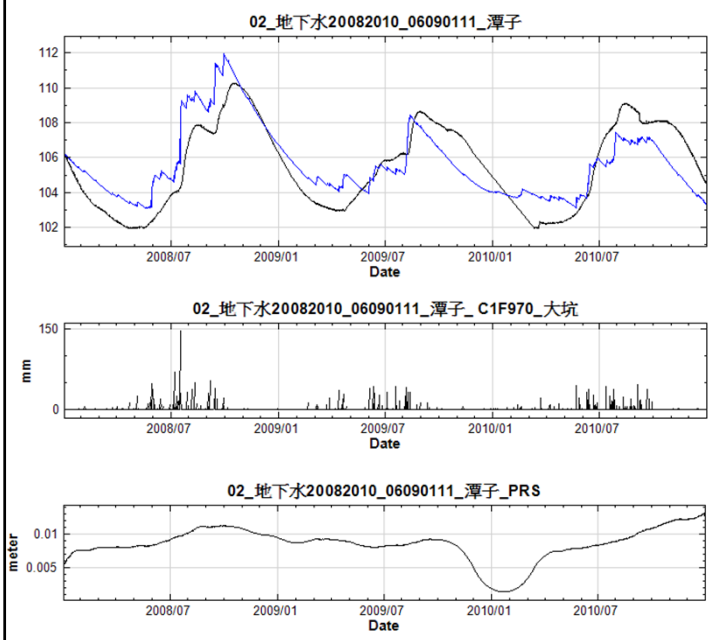

地下水測站：潭子
雨量站：大坑

λ =1.7998 (1/year)
γ =131. 7732
σ = 2.2271
u_b = 101.7956 (m)

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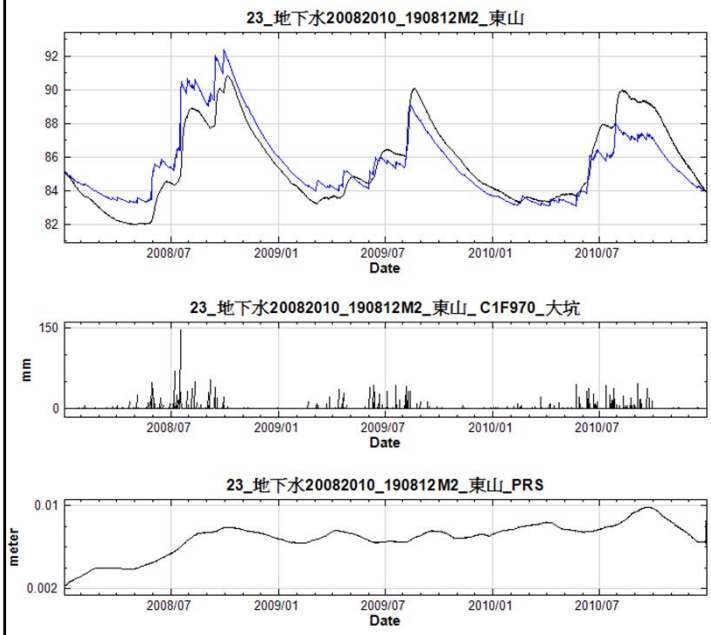

地下水測站：東山
雨量站：大坑

λ =3.5951 (1/year)
γ =150.5460
σ = 0.7957
u_b = 81.9600 (m)

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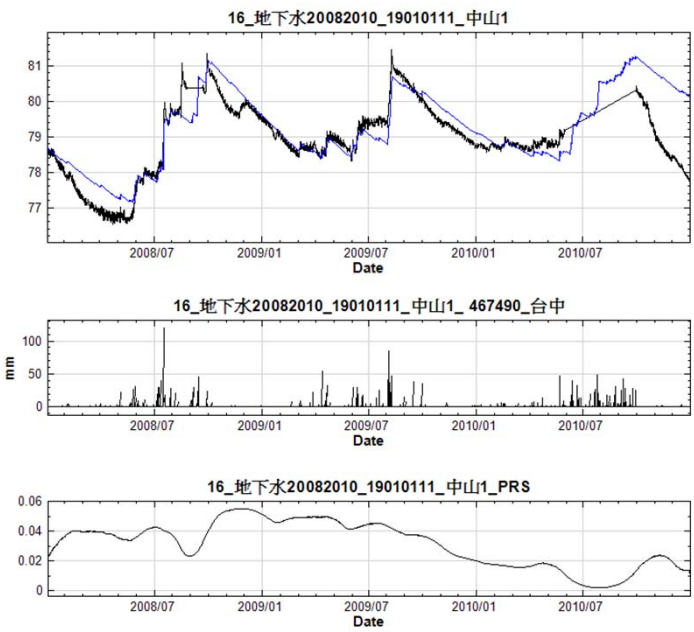

地下水測站：中山(1)
雨量站：台中

λ =1.7998 (1/year)
γ =131. 7732
σ = 2.2271
u_b = 101.7956 (m)

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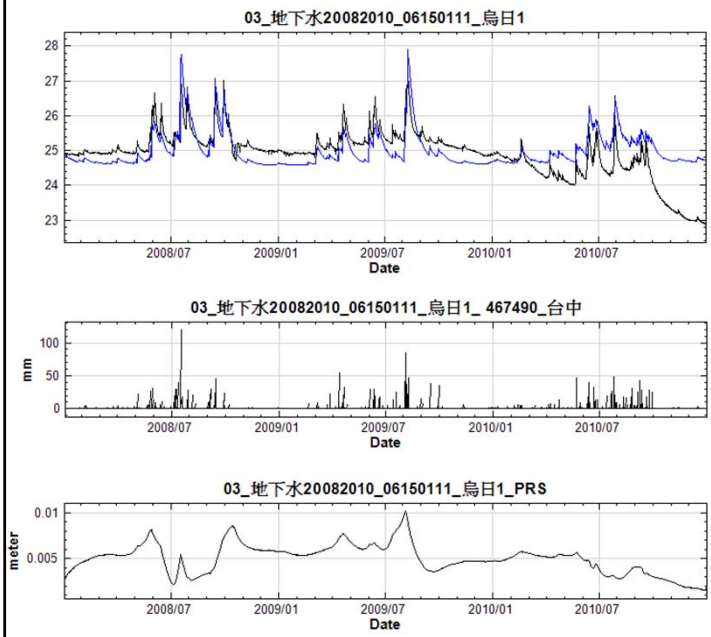

地下水測站：烏日(1)
雨量站：台中

λ =29.7439 (1/year)
γ =126.5230
σ =1.7907
u_b = 24.7100(m)

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地下水位觀測井	雨量站	λ	γ	σ	u_b
06080111_豐洲	C1F920_石岡	1.2878	199.7301	1.7032	141.3143
06090111_潭子	C1F970_大坑	1.7998	131.7732	2.2271	101.7956
06150111_烏日(1)	467490_台中	29.7439	126.5230	1.7907	24.7100
06180111_霧峰(1)	C1F9B0_桐林	3.2775	39.3707	0.0300	43.0647
061812M2_萬豐	C1F9B0_桐林	4.5932	170.8110	0.5730	53.9700
06190111_建平	C1F9A0_中竹林	8.0773	227.9148	0.1000	61.7471
06200111_大里(1)	C1F9A0_中竹林	7.2180	131.4693	0.4637	53.6000
08030111_草屯	01H630_草屯	1.9252	108.4118	3.0808	58.7636
08060111_新光(1)	01H590_六分寮	1.4093	276.6854	0.1112	160.0000
08060311_新街(1)	01H590_六分寮	0.4360	61.2684	1.3750	100.0000
19010111_中山(1)	467490_台中	0.2776	63.7419	0.1442	66.2134
19060111_啟聰(1)	C1F000_大肚	0.0365	110.3330	2.4796	72.0600
19060211_泰安	467490_台中	0.0108	118.6367	1.0000	100.1600
19080111_四張犁(1)	467490_台中	2.4520	144.8338	1.9520	94.3345
190812M2_東山	C1F970_大坑	3.5951	150.5460	0.7957	81.9600

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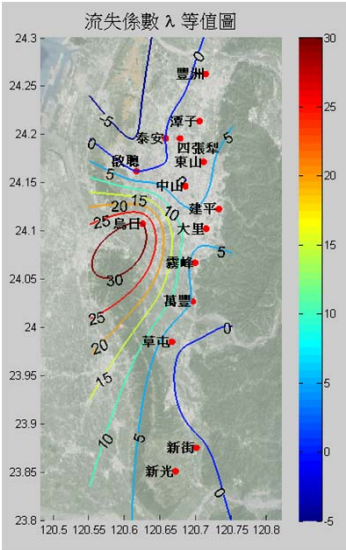
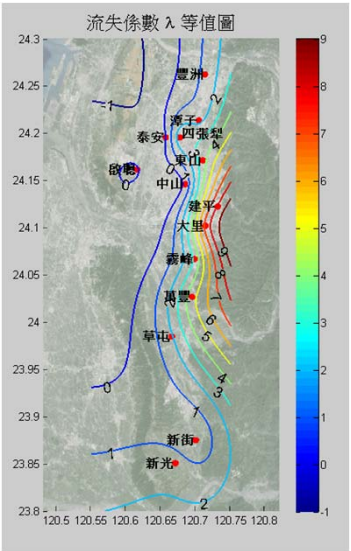
Contour of λ



（含烏日）



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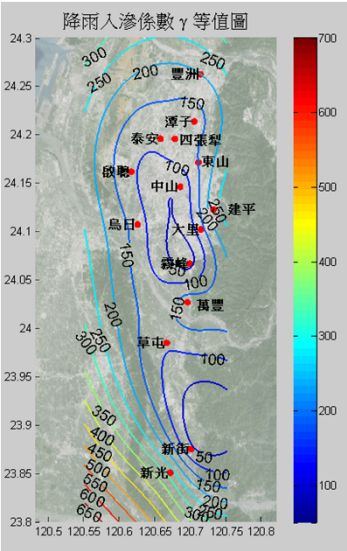
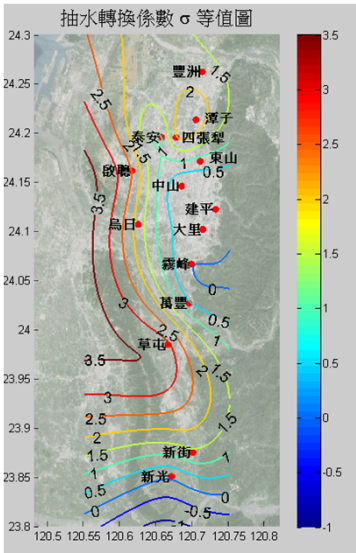
Contour of γ



Contour of σ



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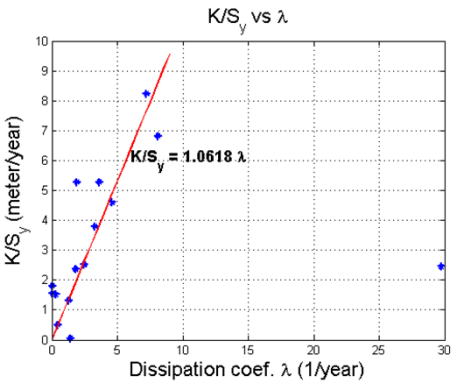
流失係數λ vs. K/Sy

地下水位觀測井	λ	S/Sy	K
06080111_豐洲	1.2878	0.1	48
06090111_潭子	1.7998	0.1	86.4
06150111_烏日(1)	29.7439	0.1	89.424
06180111_霧峰(1)	3.2775	0.1	138.38
061812M2_萬豐	4.5932	0.23	385.63
06190111_建平	8.0773	0.12	298.66
06200111_大里(1)	7.2180	0.12	360.43
08030111_草屯	1.9252	0.2	385.63
08060111_新光(1)	1.4093	0.1	1.73
08060311_新街(1)	0.4360	0.1	18.6
19010111_中山(1)	0.2776	0.13	72
19060111_啟聰(1)	0.0365	0.15	84.96
19060211_泰安	0.0108	0.14	91.87
19080111_四張犁(1	2.4520	0.1	91.87
190812M2_東山	3.5951	0.1	192.5



$$\lambda \sim \frac{K}{S}$$

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Water Balance Analysis

- 由地下水訊號模型求出：
  - 降雨入滲量
  - 抽水量
  - 流失量
  - 儲水變化量
  - 淨補注量

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Water Balance Analysis

$$\frac{\partial u}{\partial t} = -\lambda u(t) + \gamma r(t) - \sigma p(t)$$

- 取任兩個時間點積分：

$$u_2 - u_1 = -\lambda(U_2 - U_1) + \gamma(R_2 - R_1) - \sigma(P_2 - P_1)$$

$$\Delta u = -\lambda \Delta U + \gamma \Delta R - \sigma \Delta P$$

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Water Balance Analysis

$$\Delta u = -\lambda \Delta U + \gamma \Delta R - \sigma \Delta P$$

A：控制面積
S：儲水係數

$$\Delta u \times A \times S = (-\lambda \Delta U + \gamma \Delta R - \sigma \Delta P) \times A \times S$$

$$\gamma \Delta R \times A \times S = (\lambda \Delta U + \sigma \Delta P + \Delta u) \times A \times S$$

$\Delta u \times A \times S$ ：儲水變化量
$\lambda \Delta U \times A \times S$ ：流失量
$\gamma \Delta R \times A \times S$ ：降雨入滲量
$\sigma \Delta P \times A \times S$ ：抽水量

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Water Balance Analysis

$$\gamma \Delta R \times A \times S = (\lambda \Delta U + \sigma \Delta P + \Delta u) \times A \times S$$

- 降雨入滲量 = 流失量+抽水量+儲水變化量
- 淨補注量 = 降雨入滲量－流失量
  = 抽水量+儲水變化量

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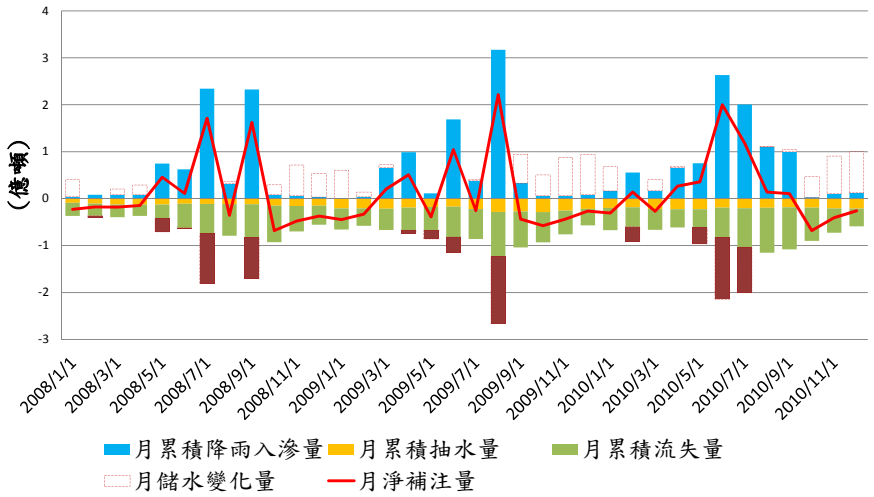
台中盆地水平衡分析



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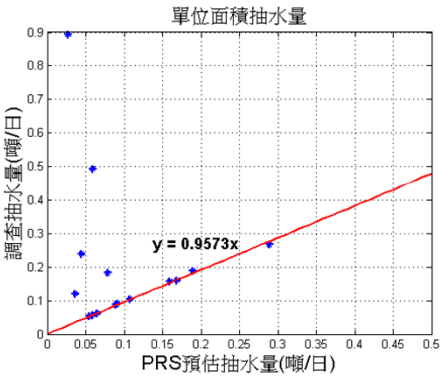
每日單位面積抽水量預測

地下水位觀測井	σ	預估	調查
06080111_豐洲	1.7032	0.1685	0.1607
06090111_潭子	2.2271	0.1893	0.1886
06150111_烏日(1)	1.7907	0.0893	0.0871
06180111_霧峰(1)	0.0300	0.0366	0.1204
061812M2_萬豐	0.5730	0.0651	0.0633
06190111_建平	0.1000	0.0270	0.8934
06200111_大里(1)	0.4637	0.0919	0.0911
08030111_草屯	3.0808	0.1592	0.1577
08060111_新光(1)	0.1112	0.1080	0.1046
08060311_新街(1)	1.3750	0.0786	0.1826
19010111_中山(1)	0.1442	0.0589	0.4926
19060111_啟聰(1)	2.4796	0.2888	0.2666
19060211_泰安	1.0000	0.0443	0.2399
19080111_四張犁(1)	1.9520	0.0597	0.0579
190812M2_東山	0.7957	0.0547	0.0535



單位：噸/日

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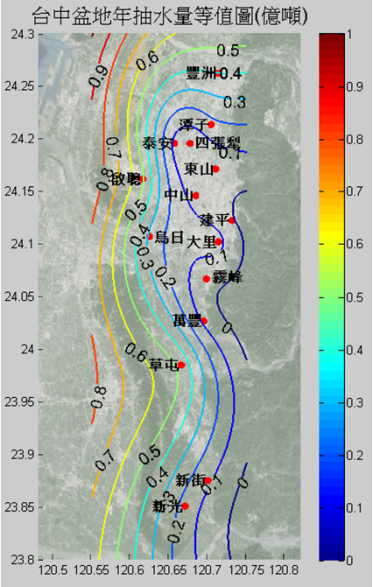

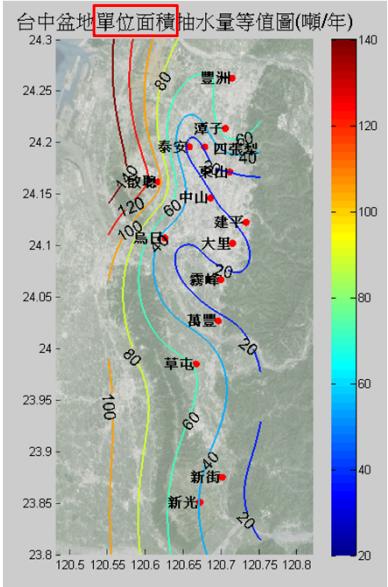

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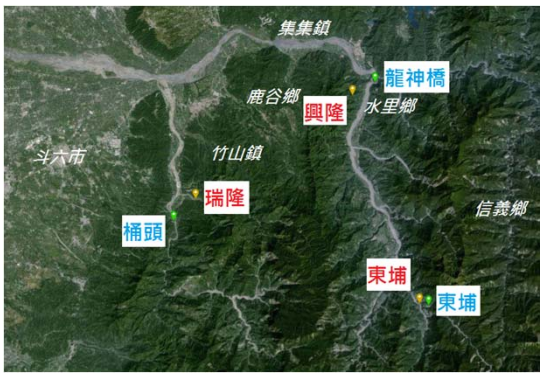
Groundwater in Mountain Area



地下水位觀測井	雨量站
BH(W)-09 興隆(淺)(深)	00H540_龍神橋
01H110_桶頭(2)	01H110_桶頭(2)
BH-26 東埔(淺)(深)	01H400_東埔

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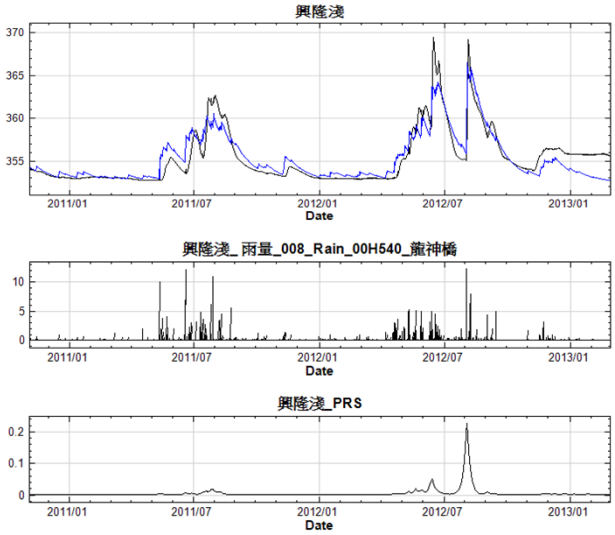
Groundwater in Mountain Area



地下水測站：興隆(淺)
雨量站：龍神橋

λ =13.2567(1/year)
γ =454.6558
σ = 0.8735
u_b = 352.5 (m)

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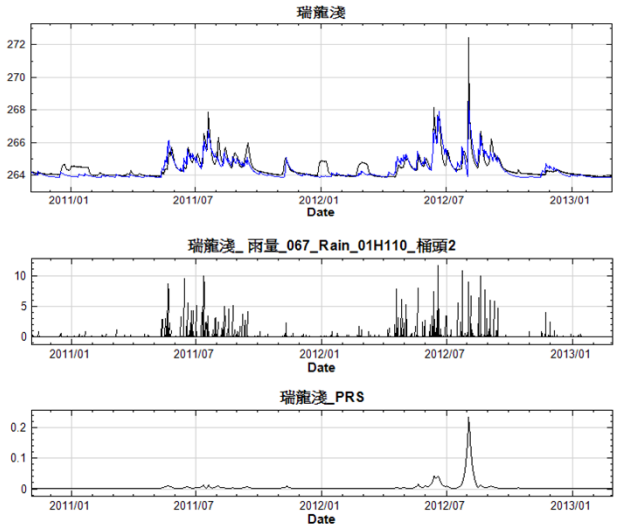
Groundwater in Mountain Area



地下水測站：瑞隆(淺)
雨量站：桶頭(2)

λ =41.6633 (1/year)
γ =1310.4
σ =1.7458
u_b = 263.9 (m)

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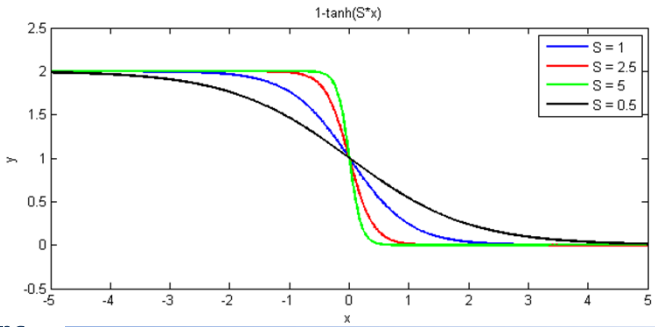
Top Level of Aquifer

$$\frac{\partial u}{\partial t} = -\lambda(u(t) - u_b) + \gamma r(t) - \sigma p(t)$$

$$\gamma(u) = \gamma_0(1 - tanh(S(\frac{u(t)}{u_{\max}} - 1)))$$



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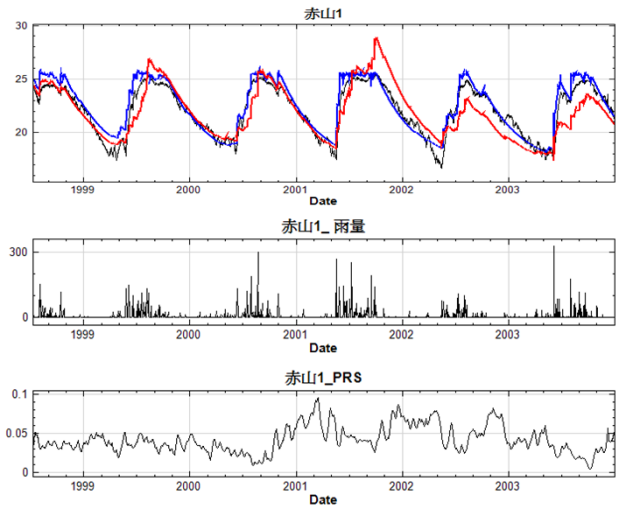
Top Level of Aquifer



地下水測站：赤山(1)
雨量站：泰武

$\lambda = 2.6665$ (1/year)
$\gamma_0 = 5.0934$
$\sigma = 0.005$
$u_b = 16.6400$ (m)

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Conclusion

- Conceptual signal modeling can briefly describe the aquifer system instead of complex partial differential equation on early estimating.
- Parameters introduced in signal modeling characterizes behavior of lumped groundwater system.
- Water balance can be obtained without physically solving partial differential equation.
- The total volume of water pumped for an area can be evaluated via PRS.

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Thank you for listening!

Q & A

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Unconfined Aquifer System Analysis

- Since the area of flow passing through would be different in unconfined aquifer, the flow rate q should be written as

$$q = -K\frac{\partial h}{\partial x}h \quad \text{(Darcy's law)}$$

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Unconfined Aquifer System Analysis

- Consider the conservation of mass:

$$\frac{\partial V}{\partial t} = S_y\frac{\partial h}{\partial t} = -\frac{\partial q}{\partial x}$$

  – Where V is the volume and S_y is the specific storage coefficient.

- Combined with the Darcy's law, then

$$S_y\frac{\partial h}{\partial t} = K\frac{\partial}{\partial x}h\frac{\partial h}{\partial x}$$

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Unconfined Aquifer System Analysis

$$S_y \frac{\partial h}{\partial t} = K \frac{\partial}{\partial x} h \frac{\partial h}{\partial x}$$

Let $h = u(t)w(x)$

$$S_y u'(t)w(x) = K u^2(t)(w'(x)w(x))'$$

$$\frac{u'(t)}{u^2(t)} = \frac{K}{S_y} \frac{(w'(x)w(x))'}{w(x)} = -\lambda$$

Then, $u'(t) = -\lambda u^2(t)$ $\boxed{u(t) \sim \frac{1}{t}}$

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Computing total volume of water pumped for an area via PRS

Since the effective distributed pumping strength can be calculated via observant PRS with known storage coefficient, overall pumping volume can be obtained by integrating the pumping strength all over the area. With discrete observant groundwater stations, such integration is reduced to summation of pumping strength times effective area. Voroini diagram can be used to define control area associated with a specific observant well. Its area is then used as the summation weighting.

Determination of confined or unconfined aquifer as well as accurate measurement of storage coefficient are crucial in the estimation.

$$Q = \int q(t)dtdA$$
$$= \sum_{k}^{n} 2\hat{q}(\mathbf{r}_k, \omega)A_k$$
$$= 2j\omega \sum_{k=1}^{n} \hat{s}(\mathbf{r}_k)S(\mathbf{r}_k)A_k$$

where

$$\hat{q} = j\omega S\hat{s}$$

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Aquifer System Analysis

- From Darcy's law, the flow rate Q can then be expressed as the gradient of piezometric head at water table.

$$Q = -KA\frac{\partial h}{\partial x}$$

  – Where K is conductivity and A is the area of flow passing through.
  – Assume flow direction is in x-axis and is constant.

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Aquifer System Analysis

- From Bernoulli's principle, the energy conservation equation of incompressible flow is

$$\rho g z + \frac{1}{2}\rho v^2 + p = \text{const.}$$

  – Where ρ is the density of the fluid, g is the acceleration due to gravity, z is the elevation of the point above a reference plane, v is the fluid flow speed, and p is the pressure.

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Aquifer System Analysis

- Since the flow speed of the aquifer is slow, the piezometric head can be written as

$$h = z + \frac{p}{\rho g} = z + \frac{p}{\gamma}$$

  – p is constant in unconfined aquifer, where the top is at atmospheric pressure , and γ is the specific weight.

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Dupuit Assumptions

- Solution for unconfined aquifer is complicated by the fact that water table varies as water is withdrawn or injected.
- Under circumstances where vertical flow of aquifer is small, Dupuit assumption reduces the nonlinear groundwater to linear one.

$$\frac{\partial p}{\partial z} = -\rho g \qquad \frac{\partial h}{\partial z} = 0$$

Vertical pressure gradient Vertical hydraulic gradient

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Dupuit Assumptions

- Flow is horizontal only. No vertical flow is allowed.
- Darcy's velocity is constant over the depth of the flow.
- Flow is under hydrostatic pressure along the vertical line.
- Spatial variation of flow is small compared to flow thickness.

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Confined/Unconfined Aquifer System

- Consider the conservation of mass:

$$\frac{\partial V}{\partial t} = S\frac{\partial h}{\partial t} = -\frac{\partial q}{\partial x}$$

  – Where V is the volume and S (or S_y) is the storage coefficient (or specific storage coefficient) for confined(or unconfined) aquifer.
- Combined with the Darcy's law, then

$$S\frac{\partial h}{\partial t} = K\frac{\partial^2 h}{\partial x^2}$$

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Confined/Unconfined Aquifer System

$$S\frac{\partial h}{\partial t} = K\frac{\partial^2 h}{\partial x^2}$$

Let $h = u(t)w(x)$

$$Su'(t)w(x) = Ku(t)w''(x)$$

$$\frac{u'(t)}{u(t)} = \frac{K}{S}\frac{w''(x)}{w(x)} = -\lambda \qquad \lambda \sim \frac{K}{S}$$

Then, $u'(t) = -\lambda u$ $\qquad \boxed{u(t) \sim e^{-\lambda t}}$

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The Difficulties of Analysis

- Accurate modeling is not possible due to inhomogeneity and anisotropic variation of hydraulic constants and unknown behavior of artificial pumping.
- Try to evaluate the change of groundwater via signal processing with lumped system assumption.

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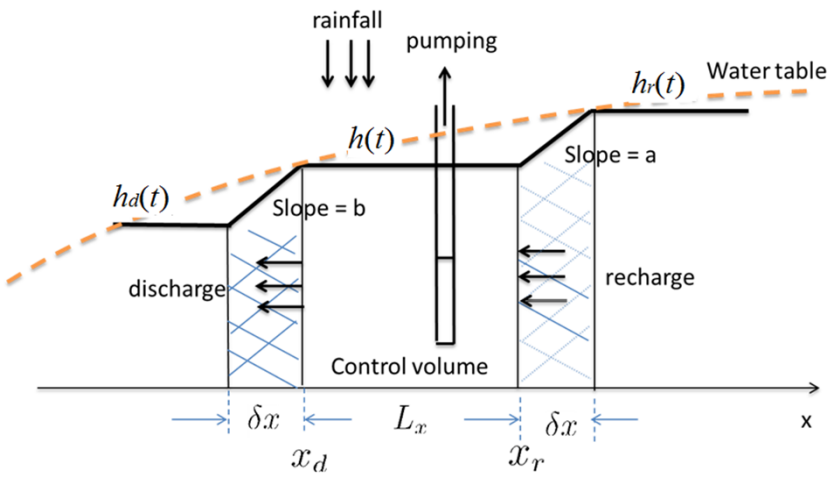
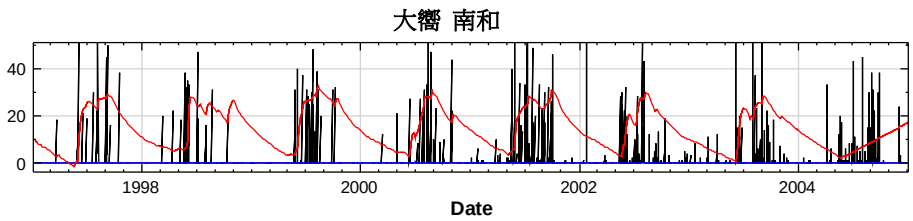
Lumped System(confined aquifer)



Schematic illustration of size specific lumped system.

Lumped Systems of Groundwater



- The level will gradually decay in dry season.
- The variation of groundwater level is highly related to rainfall.